

937e

MATHEMATICS.

With Eleven COPPER-PLATES.

By the late *Rev. Mr. Wm. WEST*
of *EXETER.*

Revised by **JOHN ROWE.**

L O N D O N :

Printed by J. KIPPAX in *Cullum-Street* ;
And sold, for the Benefit of the Author's
WIDOW, by J. RICHARDSON in *Pater-noster-*
row ; C. BATHURST in *Fleet-street* ; and
FLETCHER and Co. in *St. Paul's Church-yard.*
MDCCLXII.

MATHEMATICS

With Engraved Copper-Plates.

By the late



Revised by JOHN W. E.

3 Hh

LONDON:

Printed by J. KIPAX in Pall-mall Street:
And sold for the Benefit of the Author's
Widow, by J. Richardson in Fleet-street:
and
new; C. Bathurst in Fleet-street; and
Fletcher and Co. in St. Paul's Church-yard.
MDCCLXII.

A P O L O G Y.

THE late ingenious and learned Author having wrote and left behind him several mathematical Papers; the Editor, for the Benefit of the Widow, was desired to select, and prepare for the Press, *all* such as he might think would meet with a favorable Reception from the Public: And these selected, he hopes, will not dishonor the Memory of his worthy Friend: — He has altered no more than what appeared to him to be absolutely necessary: But some Lines he was obliged to interweave; for few of the Papers were finished, and none, perhaps, ever intended for public Inspection. — He depends on the Candor and good Sense of the Reader; and thinks it impertinent to point out those Parts which he imagines will be most approved. — Some Things the Editor has taken the Liberty to insert *as his own*; for which, he hopes, he shall rather be kindly excused than severely censured.

April 20,
1761.

JOHN ROWE,

A P O L O G Y.

THE late ingenious and learned Author having wrote and left behind him several mathematical Papers; the Editor, for the Benefit of the Widow, was desir'd to select, and prepare for the Press, all such as he might think would meet with a favorable Reception from the Public: And thus select'd, he hopes, will

— He has alter'd no more than what appeared to him to be absolutely necessary: But some

✶ The Author died October 1, Anno 1760,

✶ Aetat. aet. 53.

the Papers were published, and none, perhaps, ever intended for public Instruction. — He de-

Reader; and thinks it impertinent to point out those Parts which he imagines will be most ap-

proved. — Some Things the Editor has taken

the Liberty to insert as his own; for which, he

hopes, he shall rather be kindly excus'd than

severely censur'd.

John Rowe.

April 20.
1761.

SUBSCRIBERS.

A.

MR. Thomas Abney, Birmingham
Mr. John Appleby, Cannon-row

B.

Mr. Joseph Barker, Surgeon, London
Mr. John Barrell, Merchant, London
Rev. Mr. Bartlett, Sidbury
Mr. Bennett, Crown-court
John Bevis, M. D. Member of the R. A. S. at Berlin
Thomas Birch, S. T. P. & R. S. Sec.
Rev. John Blair, L. L. D. & R. S. S. 6 Copies
Patrick Blair, M. D. Cork
Mr. Ant. Blinkhorne of the Custom-house
Rev. Mr. Bourn, Norwich, 6 Copies
Mr. Joseph Bretland, jun. Exeter
Mr. Arthur Brown, Honiton
Mr. John Veryard Brutton of Sidney College,
[Cambridge]

C.

John Canton, M. A. & F. R. S.
Mr. Wm. Carpenter, Sarum
Mr. Wm. Chapple, Exeter, 2 Copies
John Coe, Esq. Maldon
Mr. Collier, Topham [Woolwich]
Mr. Cowley, Master of the Royal Academy at
Mr. John Crang, Organ-BUILDER, Wych-street
Mr. Geo. Dance, City Surveyor
Mr. James Davies, Carmarthen
Mr. Wm. Dickes, London
Mr. Rich. Dixon, Carpenter, Air-street, 2 Copies
Mr. Joseph Dixon, Mason, St. Alban's-street, 2
Mr. Benj. Dohm, Bideford, 2 Copies [Copies
Mr. Peter Dupont, Watch-maker, Ivey-lane
Mr. Dyer, Moreton-Hampstead

SUBSCRIBERS.

Rev. Mr. Edwards, Tisbury
 Mr. W. E. 3 Copies
 Mr. Samuel Eveleigh, Bristol
 Mr. Samuel Follett, Sidmouth
 J. F. D. D.
 H. [the Bank]
 Mr. Nath. Hammond, Accountant-General of
 Mr. Harris, Mathematical Master at Christ's
 Hospital, 2 Copies
 Rev. Mr. Harris, Honiton, 6 Copies
 Rev. Mr. Harrison, Taunton
 Rev. Mr. Hewgoe, Rector of Dunchidock
 Mr. Hill, Chancery-lane
 Rev. Mr. Hogg, Sidmouth
 Rev. Mr. Holden, Maldon
 Rich. Hole, Esq. Exeter
 Tho. Hollis, Esq. F. R. S. 5 Copies
 Rev. Mr. Hoeke, Branscomb
 Barth. Jeffery, Esq. Exeter
 Mr. Geo. Jeffery, Merchant, London, 6 Copies
 Rev. Mr. Joseph Jeffries, London
 Rev. Mr. Jenkins, Tutor at the Academy in
 Mr. Wm. Irwin, Appledore [Carmarthen]
 Rev. Mr. John Kiddell, Tiverton
 Mr. King, Honiton
 Mr. John Kittle, Birmingham
 Mr. Israel Llewellyn, Carmarthen
 Mr. Joseph Mahoon, Harpsichord-maker to his
 Mr. John Mallett, Exeter, 6 Copies [Majesty
 Mr. B. M.

SUBSCRIBERS.

Mr. F. M.

Mathematical Society, St. James's, 4 Copies

Mr. Wm. Meyler, Carmarthen

John Mills, Esq. London

Rev. Mr. Morgan, Cullumpton

Mr. Tho. Moss, London

Nicolas Munckley, Esq. Lincoln's-Inn

Rev. Mr. Murthwaite

Mr. Rob. Mylne, Architect, and Surveyor to
the New Bridge at Black-Friars

Mr. James Mynde, F. R. S.

N.

Mr. Kellow Nation, Exeter

Mr. Tho. Newberry, Professor of Mathematics
and Navigation in the Noble Cadet Corps
of Marines at St. Petersburg

P.

Mr. Wm. Parr, Moreton-Hampstead

Rev. Mr. Parry, Cirencester, 2 Copies

John Stocker Pearse, Esq. Sidbury

Mr. Wm. Penny, Surgeon, Newton-Abbot

Mr. Wm. Perkins, Merchant, London

Mess. Powell and Layton, Borough of Southwark,

Mr. Wm. Marker Powell, London [2 Copies

Rev. Mr. Price, Newington

Rev. Mr. William Prior, London

R.

Mr. Wm. Reeves, Master of the Academy in
Bishopsgate-street, 6 Copies

Mr. Ald. Richardson, Durham, 6 Copies

Mr. Edward Rollinson, Duke-street

Mr. Joseph Rosser, Bristol

Mr. Jacob Rowe, Exeter

J. R. Throgmorton-street, 6 Copies

Mr. Edward Rush, Mafon, Batterlea

S,

SUBSCRIBERS.

S.

Mr. Tho. Sansum, Carpenter, Hastings
 Rev. Mr. Sam. Morton Savage
 Mr. Edw. Score, Exeter, 6 Copies
 Mr. Rob. Shirtcliffe, London
 Mr. Geo. Silverfide, Royal Bagnio in Newgate-
 Mr. John Simpson, Carpenter, St. John's, West-
 Mr. Joseph Smith, Birmingham
 Mr. Joseph Smith, Bristol
 Mr. Smith, Exeter
 Mr. John Spencer, Carpenter, Park-street, 6 Co-
 John Stephens, Esq. Exeter, 7 Copies

T.

Mr. Rob. Taylor, Architect, Spring-Gardens
 Rev. Mr. Samuel Thomas, Tutor at the Aca-
 demy in Carmarthen
 Rev. Mr. Toulmin, Culliton

Mr. Matthew Towgood, London
 Rev. Mr. Turner, Lympson

U

Mr. Upton, Attorney at Law, Winchester-street

W.

Rev. Mr. Tho. Watton, Bridgewater
 Capt. Webber, London
 Mr. James White, Friday-street
 Mr. John White, Exeter
 Rev. Mr. Williams
 Mr. Wm. Williams, Exeter
 Mr. Samuel Wilton, jun.
 Mr. Wm. Woolcombe, Rotherhithe
 Thomas Wright, Esq.

C. H. A. P.

CHAP. I.

FLUXIONS.

SECT. I.

Of the Doctrine of Fluxions.

1. **Q**UANTITIES, whose Values are mutable, are called *Fluents*, or *variable* or *flowing Quantities*: And those, whose Values are immutable, are called *fixed* or *invariable Quantities*. Thus, the Diameter of a Circle, and the Parameter of a Parabola, are *invariable Quantities*; and the Absciss and Ordinate of any Curve are *variable Quantities*.

2. Variable Quantities are supposed to be produced after the Manner of a Space which a Body in Motion describes. Thus, a flowing Line is supposed to be generated by a Point in Motion, a Surface by a Line, and a Solid by a Surface.

3. The *Velocity* with which any Quantity flows, increases, or decreases, at any particular
B Point

Point or Term, is called the *Fluxion* of that Quantity at that Point or Term.

4. The indefinitely small *Increase or Decrease* of a Quantity, effected in an indefinitely small Particle of Time, is called the *Moment, Increment, or Decrement*, of that Quantity.

5. Spaces uniformly described, being as the Velocities of the describing Bodies; and, the Time in which any Increments or Decrements are effected being indefinitely small, and consequently the Velocities with which they are generated may be considered as uniform; therefore, the *Increments or Decrements of Quantities* may be taken either as proportional to or for their *Fluxions*; and *vice versa*.

6. Variable Quantities are signified by the last Letters of the Alphabet; and invariable Quantities by the beginning.

7. A Dot set over the Head of any single Quantity, stands for the *Fluxion* of that Quantity: Thus, $\dot{x}$ represents the Fluxion of x . And, the *Moment, Increment, or Decrement*, of any Quantity x is represented by x' .

8. The Fluxion of a compound Quantity, or Product, as xy , is $= \dot{x}y + x\dot{y}$. For, (x and y being flowing and increasing Quantities,) when x becomes $x + x'$, then y will become $y + y'$; and therefore xy will become $= (x + x') \times (y + y') = xy + x'y + xy' + x'y'$, and consequently the Increment of the Fluent xy will be $= x'y + xy' + x'y'$: But, x' and y' being infinitely little in Comparison of x and y ; therefore, $x'y'$ may be rejected; and then the Increment of xy will be represented by $x'y + xy'$. Hence, (*Art. 5.*)

the

the Fluxion of xy is $\dot{x}y + x\dot{y}$. From this Method of Reasoning may easily be deduced the following RULE for finding the Fluxion of the Product of any Quantities whatever; viz. Take the Fluxion of each simple Quantity of the Product, and multiply it severally into all the rest; and add these Products together. Thus, the Fluxion of xyz is $\dot{x}yz + x\dot{y}z + xy\dot{z}$; and (because an invariable Quantity does not either increase or decrease, and consequently its Fluxion is $\dot{} = 0$.) the Fluxion of axy is $\dot{a}xy + a\dot{x}y$, and the Fluxion of $axyz$ is $\dot{a}xyz + a\dot{x}yz + ax\dot{y}z$. The same may be easily applied to several Products connected with the Signs $+$ and $-$. For Example, the Fluxion of $ab + bx - ay - xy$ is $\dot{a}b + b\dot{x} - \dot{a}y - a\dot{y} - \dot{x}y - x\dot{y}$.

9. For the Fluxion of a Fraction, the RULE is, Multiply the Fluxion of the Numerator into the Denominator, to which connect, with the Sign $-$, the Fluxion of the Denominator multiplied into the Numerator, for a new Numerator; and set down the Square of the Denominator for a new Denominator. Thus, the Fluxion of $\frac{x}{y}$ is $\frac{\dot{x}y - x\dot{y}}{y^2}$: For, suppose $\frac{x}{y} = z$; then $x = yz$, and, consequently, $\dot{x} = \dot{y}z + y\dot{z}$, and $\frac{\dot{x} - y\dot{z}}{y} = \dot{z}$ = the Fluxion of $\frac{x}{y}$ by Supposition; but, $\frac{\dot{x} - y\dot{z}}{y} = \frac{\dot{x}y - y\dot{z}y}{y^2} =$ (by substituting x for its Value yz .) $\frac{\dot{x}y - x\dot{y}}{y^2}$: therefore, &c.

ro. For the Fluxions of all Powers *universally*; suppose the Index or Exponent of any Power to be m , as x^m ; the Fluxion of that Quantity is $= mx^{m-1} \dot{x}$. To prove this, suppose first of all $m =$ an Whole-number, for Example, $= 3$; then $x^m = xxx$, whose Fluxion is $= \dot{x}xx + x\dot{x}x + xx\dot{x}$ by the Rule of Products (*Art. 8.*) *i. e.* $= 3x^2\dot{x} = mx^{m-1} \dot{x}$. Secondly, Suppose the Exponent to be negative, as $x^{-m} = \frac{1}{x^m}$: Here, the Fluxion will be $= \frac{-mx^{m-1} \dot{x}}{x^{2m}}$ (by the foregoing Rule;) *i. e.* $= -mx^{-m-1} \dot{x}$. Thirdly, suppose the Exponent to be a fractional Quantity, as $x^{\frac{m}{n}}$ (or $\sqrt[n]{x^m}$.) Here ($\frac{m}{n}$ being any Fraction,) put $x^{\frac{m}{n}} = z$; then $x^m = z^n$, and $mx^{m-1} \dot{x} = nz^{n-1} \dot{z}$, and $\frac{mx^{m-1} \dot{x}}{nz^{n-1} \dot{z}} = \dot{z} =$ the Fluxion of $x^{\frac{m}{n}}$ by the Supposition; but $nz^{n-1} \dot{z} =$

$mx^m - \frac{m}{n}$: Consequently $\frac{mx^{m-1} \dot{x}}{mx^m - \frac{m}{n}}$, that is,

$\frac{m}{n} x^{\frac{m}{n} - 1} \dot{x}$ = the Fluxion of $x^{\frac{m}{n}}$. Lastly, suppose the Exponent to be both fractional and negative,

as $x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}}$; then (*Art. 9.*) the

Fluxion will be $= \frac{-\frac{m}{n} x^{-\frac{m}{n}-1} \dot{x}}{x^{\frac{m}{n}}}$

$= -\frac{m}{n} x^{-\frac{m}{n}-1} \dot{x}$; like all the former Cases;

which shews the Rule to be *universal*; viz. Let m express any Quantity, Whole-number, or Fraction, negative or positive; then, the

Fluxion of x^m will be $= mx^{m-1} \dot{x}$; that is, Multiply the Coefficient by the Index of the Power, subtract 1 from the said Index, and then multiply by the Fluxion of the Root.

THE Uses of this Doctrine here comprised in so small a Compass, infinitely exceed the Bounds of the strongest Fancy and most copious Invention; as it may be applied to an infinite Number of Subjects of infinitely various Kinds.

SECT.

S E C T. II.

The Doctrine of Fluxions applied to the Solution of Problems de Maximis et Minimis.

11. **T**HE greatest Magnitude of a Quantity, that first increases and afterwards decreases, is called a *Maximum*; and, the least Magnitude of a Quantity, that first decreases and afterwards increases, is called a *Minimum*.

12. The common Way for finding the *Maximum* or *Minimum* of a flowing Quantity, is, to make the *Fluxion* of it $= 0$: But, as the Reason of this proceeding is not so easily apprehended, the following new Method may, perhaps, be deemed preferable to it.

Prob. 1.

13. Let the given right Line AB (Fig. 1.) be so divided at C, as that the Rectangle of the two Parts AC and CB shall be a *Maximum*, or, greater than if the said Line was divided any other where.

Common Way.

Put $AB = a$, $AC = x$, and $CB = a - x$; then $ax - x^2 = a$ Maximum by Quest. the Fluxion of which is $= 0$; that is, $ax - 2xx = 0$; therefore $a = 2x$, and $x = \frac{1}{2}a$.

S E C T.

New

Let the Point of Division, C, begin at A, and move towards B; then it is plain, that the Side AC of the Rectangle will be continually gaining; and the other Side of it, as BC, continually losing. Now, though one Side gains exactly as much as the other loses, if we consider only the Sum of the two Sides or Lines AC and CB; yet, if we consider the Rectangle or Product of them, every Increment or Increase of AC in the Beginning is vastly more than the same Decrement or Decrease of BC; because the Increases of AC are to be multiplied into BC, but the Parts of BC are only to be multiplied into AC, which is always less than BC, until the Point C reaches the Midway between A and B; thus, a Rectangle of unequal Dimensions, gaining just as much in the lesser as it loses in the larger Dimension, increases in Quantity; *e. g.* the Rectangle DCBE is more than the Rectangle FCBH, (DH being more than FC, as GH is more than GC). — Suppose now the Rectangle CIKB a Maximum, and consequently the Point C the Point required: and let AB be supposed divided into an infinite Number of small and equal Parts, each (*e. g.*) the thousandth Part of an Hair's Breadth = x . Call AC, x ; and CB, y ; then, it is plain from the above Reasoning, that, $xx = yy$, and consequently, $x = y$; as before.

N. B. The above Equation might stand thus, *viz.* $xy = yx$; but $y = x$ = the Velocity of the Point C; therefore $x = y$; or thus, As

$x : y :: \dot{x} : \dot{y} \therefore x\dot{y} = y\dot{x}$; and as $\dot{x} = \dot{y}$, therefore $x = y$.

Note, The Reverse of this Problem is, to find two Quantities whose Product is given and whose Sum is a *Minimum*.

Prob. 2.

14. Let the Line AB (Fig. 1.) be so divided, as that one Part multiplied into the Square of the other shall be a Maximum.

Common Way.

Put $AB = a$, $BC = x$, and $AC = a - x$; then $a - x \times x^2 =$ a Maximum $= ax^2 - x^3$: In Fluxions, $2ax\dot{x} - 3x^2\dot{x} = 0$. Hence, by dividing by $3x\dot{x}$, we have $\frac{2}{3}a - x = 0$; $\therefore x = \frac{2}{3}a$.

New Way.

As long as the Line AC increases faster in Proportion to its Quantity than the Square of CB decreases in Proportion to its Quantity, their Product must increase; but, as soon as the Square of BC begins to decrease faster in Proportion (as before) than the Line AC increases; then their Product begins to decrease: Consequently, that Product is a Maximum, just as the Velocity of Increase, or Fluxion of AC, is to the Fluxion of the Square of CB, as AC to the Square of CB: that is, putting $BC = x$, and $AC = y$, As $y : x^2 :: \dot{y} : 2x\dot{x}$; $\therefore x^2\dot{y} = 2xy\dot{x}$, and $x\dot{y} = 2y\dot{x}$; that is, As $\dot{y} : \dot{x} :: 2y$

$\therefore 2y : x$; but $y = x$; for, whatever be the Increment of AC, that same must be the Decrement of BC: Consequently $2y = x$, as before; for, if $2y = x$, then $3y = a$, and $x = \frac{2}{3}a$.

Note. The Reverse of the foregoing Problem is this, *viz.* Given the Product of one Quantity multiplied into the Square of another; required these two Quantities when their Sum is a *Minimum*: that is, given $yx^2 = a$; required y and x , when $y + x = a$ *Minimum*.

Prob. 3.

15. To render the foregoing Problem more general: — Let it be required to divide a given right Line so as that one Part drawn into the n th Power of the other Part shall be a Maximum.

Common Way.

Put AB (*Eig. 1.*) $= a$, BC $= x$, and AC $= a - x$; then $(a - x) \times x^n = ax^n - x^{n+1} = a$ Maximum; the Fluxion of which is $nax^{n-1} \dot{x} - (n+1)x^n \dot{x} = 0$; therefore (by dividing by $x^{n-1} \dot{x}$) $na = (n+1)x$, and $x = \frac{n}{n+1}a$.

Whence, if $n=1$, then $x = \frac{1}{2}a$; if $n=2$, then $x = \frac{2}{3}a$; if $n=3$, then $x = \frac{3}{4}a$, &c.

New Way.

According to the new Method of Reasoning, before laid down, the Fluxion of AC must be to the

the Fluxion of the n th Power of CB as the Quantity AC to the n th Power of CB; that is, (putting $BC=x$, $AC=y$, and $AB=a$,) As y :

$x^n :: y : nx^{n-1}\dot{x}$; Consequently, $x^n y = ynx^{n-1}\dot{x}$,

and (dividing by x^{n-1}) $x y = y n \dot{x}$; but $y = \dot{x}$, $\therefore$

$x = ny$, and $y = \frac{x}{n}$. Now, $x+y=a$, therefore

$$x + \frac{x}{n} = a, \text{ and } nx + x = na, \text{ and } x = \frac{n}{n+1} a;$$

as before.

Prob. 4.

16. *The preceding Problem will be absolutely universal, if it be required to divide a Line so, as that the Product of the m th Power of one Part multiplied into the n th Power of the other, may be a Maximum.*

Common Way.

Put AB (Fig. 1.) $=a$, $BC=x$, and $AC=a-x$; then, $(a-x)^m \times x^n =$ a Maximum, by

the Quest. so that, $nx^{n-1}\dot{x} \times (a-x)^m -$

$m(a-x)^{m-1}\dot{x} \times x^n = 0$; that is, (dividing by

$x^{n-1}\dot{x}$) $n(a-x)^m = mx(a-x)^{m-1}$, and (dividing

by $(a-x)^{m-1}$) $na - nx = mx$, i. e. $na - nx = mx$,

or $na = mx + nx$; $\therefore x = \frac{n}{m+n} a$. Whence, if

$$m=n,$$

$m = n$, then $x = \frac{1}{2}a$, if $m = 1$ and $n = 2$, then $x = \frac{2}{3}a$, &c. as before.

New Way.

Put $AB = a$, $BC = x$, and $AC = y$; then, according to the new Method pursued in the foregoing Problems, As $y^m : x^n :: my^{m-1}j : nx^{n-1}\dot{x}$. Consequently, $x^n my^{m-1}j = y^n nx^{n-1}\dot{x}$; and, as $j = \dot{x}$, so (dividing by $x^{n-1}y^{m-1}$, and throwing j and $\dot{x}$ out,) $xm = yn$, and $y = \frac{m}{n}x$; and, as $x + y = a = x + \frac{m}{n}x$, so $na = nx + mx$, and $x = \frac{n}{m+n}a$; as before.

Corollary. The Parts in Quest. viz. AC and BC, are in direct Proportion to their Powers, m and n , in the Maximum; (i. e.) $AC : CB :: m : n$.

Prob. 5.

17. Required the Point C in the Diameter AB of a Circle, through which the perpendicular Chord DE being drawn, the Product of AC into DE shall be a Maximum. (Fig. 2.)

Common Way.

Call AB, a ; AC, x ; BC, $a - x$; CE, y ; and DE, $2y$. Then $ax - x^2 = y^2$, and $4ax - 4x^2 = 4y^2 = \overline{DE}^2$, and $\sqrt{4ax - 4x^2}^{\frac{1}{2}} = DE$; so
C 2
that

that $x \times \sqrt{4ax - 4x^2}^{\frac{1}{2}} = \text{a Maximum}$. In Fluxions, $\sqrt{4ax - 4x^2}^{\frac{1}{2}} \dot{x} + \frac{1}{2}x \times \sqrt{4ax - 4x^2}^{-\frac{1}{2}} \times 4a\dot{x} - 8x\dot{x} = \sqrt{4ax - 4x^2}^{\frac{1}{2}} \dot{x} + \frac{2ax\dot{x} - 4x^2\dot{x}}{\sqrt{4ax - 4x^2}^{\frac{1}{2}}} = 0$;

which (multiplying by $\sqrt{4ax - 4x^2}^{\frac{1}{2}}$) gives $4ax - 4x^2 + 2ax - 4x^2 = 0$, or, $6ax = 8x^2$, $\therefore x = \frac{3}{4}a = \frac{3}{4}a = \text{AC required}$.

New Way.

Put $AB = a$, $AC = x$, and $BC = y$. Now, according to the Reasoning proposed and pursued in the foregoing Problems; the Product of AC into DE is a Maximum just when the Fluxion of AC is to the Fluxion of DE as AC is to DE; that is, (because $\sqrt{4xy}^{\frac{1}{2}} = DE$) As $x : \sqrt{4xy}^{\frac{1}{2}} :: \dot{x} : (\frac{1}{2} \times \sqrt{4xy}^{-\frac{1}{2}} \times 4x\dot{y} - 4y\dot{x} =) \frac{2x\dot{y} - 2y\dot{x}}{\sqrt{4xy}^{\frac{1}{2}}}$; which gives $\sqrt{4xy}^{\frac{1}{2}} \dot{x} = \frac{2x^2\dot{y} - 2yxx\dot{x}}{\sqrt{4xy}^{\frac{1}{2}}}$;

and multiplying by $\sqrt{4xy}^{\frac{1}{2}}$, we have $4xy\dot{x} = 2x^2\dot{y} - 2yxx\dot{x}$; and, as $\dot{x} = \dot{y}$, so $4xy = 2x^2 - 2xy$, and $6xy = 2x^2$, and $3y = x$. Conseq. $y = \frac{1}{3}a$, and $x = \frac{2}{3}a$; as before.

* *N. B.* As to the Quantities x and y ; when one is increasing the other is decreasing; and therefore their Fluxions must be affected with contrary Signs when on the same Side of the Equation.

Note.

Note. By this Problem, the largest Triangle may be inscribed in a Circle; A, D, and E, being the angular Points.

Prob. 6.

18. Required the Point C in the Diameter AB of a Circle, through which the perpendicular Chord DE being drawn, the Product of AC into the n th Power of DE may be a Maximum. (Fig. 2.)

Common Way.

Put $AB=a$, $AC=x$, and $BC=a-x$. Then,
 $x \times \sqrt[n]{4ax-4x^2} = \text{a Maximum by the Quest.}$

The Fluxion of which is $\sqrt[n]{4ax-4x^2} \dot{x} + x^{\frac{1}{n}} \frac{1}{n} \cdot$

$\frac{4a-4x}{\sqrt[n]{4ax-4x^2}^{n-1}} \times \sqrt[n]{4ax-4x^2} \dot{x} = \sqrt[n]{4ax-4x^2} \dot{x} +$

$nx \cdot \sqrt[n]{4ax-4x^2}^{\frac{1}{n}-1} \times 2a\dot{x}-4x\dot{x} = 0$; which

divided by $\sqrt[n]{4ax-4x^2}^{\frac{1}{n}-1}$, gives $\sqrt[n]{4ax-4x^2} \cdot \dot{x}$

$+ nx \cdot 2a\dot{x}-4x\dot{x} = 0$, and (dividing by $\dot{x}$) $4ax$

$-4x^2+2nax-4nx^2=0$, $\therefore 4ax+2nax=4x^2+$

$4nx^2$, that is, $4+2n \cdot ax=4+4n \cdot x^2$, or (di-

viding by x) $4+2n \cdot a=4+4n \cdot x$; $\therefore x=$

$\frac{2+n}{2+2n} a$. So that, if $n=1$, then $x=\frac{3}{4}a$, as be-

fore in Prob. 5. And if $n=2$, then $x=\frac{3}{4}a$;

in which Case it would be a Canon for inscribing

the largest Cone in a given Sphere: For, if the

Circle ADBE revolve upon its Axis AB, it will

describe a Sphere, and the Triangle inscribed in

it

it will describe a Cone. Now, Cones are as the Products, viz. of the perpendicular Height into the Square of the Diameter of the Base; that is, as $AC \times DE^2$.

New Way.

Put $AB = a$, $AC = x$, and $BC = y$: Then $\overline{4xy}^{\frac{1}{2}} = DE$. Now, as $x : \overline{4xy}^{\frac{1}{2}} :: \dot{x} : \frac{1}{2}n \cdot \overline{4xy}^{\frac{1}{2}n-1} \times \overline{4xy} - 4\dot{x}y$; whence $\overline{4xy}^{\frac{1}{2}n} \dot{x} = \frac{1}{2}nx \cdot \overline{4xy}^{\frac{1}{2}n-1} \times \overline{4xy} - 4\dot{x}y$; that is, (dividing by $\overline{4xy}^{\frac{1}{2}n-1}$), $4xy\dot{x} = nx \cdot \overline{4xy} - 4\dot{x}y$; and as $\dot{x}$ and $\dot{y}$ are equal (as they stand precisely for one and the same Quantity,) it will be $4xy = 2nx^2 - 2nxy$, or, $4y + 2ny = 2nx$; $\therefore y = \frac{n}{2+n} x$; where, if $n=1$, then $y = \frac{1}{3}x$, and $x + \frac{1}{3}x = \frac{4}{3}x = a$, and $x = \frac{3}{4}a$. If $n=2$, then $y = \frac{2}{5}x$, and $x = \frac{5}{7}a$, as before.

Prob. 7.

19. To render the foregoing Problem universal: — Let it be required to find the Point C in the Diameter AB, through which the perpendicular Chord DE being drawn, the Product of the m th Power of AC multiplied into the n th Power of DE shall be a Maximum. (Fig. 2.)

Common

Common Way.

Put $AB=a$, $AC=x$, and $BC=a-x$; then
 $x^m \times \overline{4ax-4x^2}^{\frac{1}{2}n} = a$ Maximum, by Quest.
 In Fluxions, $\frac{1}{2}nx^m \cdot \overline{4ax-4x^2}^{\frac{1}{2}n-1} \times 4a\dot{x}-8x\dot{x}$
 $+ \overline{4ax-4x^2}^{\frac{1}{2}n} \times mx^{m-1}\dot{x} = 0$, or, $nx^m \times$
 $\overline{4ax-4x^2}^{\frac{1}{2}n-1} \times 2a\dot{x}-4x\dot{x} + \overline{4ax-4x^2}^{\frac{1}{2}n} \times$
 $mx^{m-1}\dot{x} = 0$; which divided by $\overline{4ax-4x^2}^{\frac{1}{2}n-1}$
 $\times x^{m-1}$, gives $nx \cdot 2a\dot{x}-4x\dot{x} + m \cdot 4ax-4x^2 \cdot \dot{x} =$
 0 ; and this Equation divided by $x\dot{x}$ gives $2an-$
 $4nx+4am-4mx=0$; $\therefore 2n+4m \cdot a = 4n+4m \cdot x$,
 and $x = \frac{n+2m}{2n+2m} a$. Here then, if $m=n$, $x =$
 $\frac{3}{2}a$; if $m=1$, and $n=2$, then $x = \frac{2}{3}a$; as be-
 fore in Prob. 6.

New Way.

Put $AB=a$, $AC=x$, and $BC=y$; then
 $\overline{4xy}^{\frac{1}{2}} = DE$, and $x^m \times \overline{4xy}^{\frac{1}{2}n} = a$ Maxi-
 mum, by Quest. Now, As $x^m : \overline{4xy}^{\frac{1}{2}n} ::$
 $mx^{m-1}\dot{x} : (\frac{1}{2}n \cdot \overline{4xy}^{\frac{1}{2}n-1} \times 4x\dot{y}-4\dot{x}y =)$
 $n \cdot \overline{4xy}^{\frac{1}{2}n-1} \times 2x\dot{y}-2\dot{x}y$; which gives this E-
 quation, viz. $\overline{4xy}^{\frac{1}{2}n} \times mx^{m-1}\dot{x} = nx^m \cdot \overline{4xy}^{\frac{1}{2}n-1}$
 $\times 2x\dot{y}-2\dot{x}y$, and (dividing by $\overline{4xy}^{\frac{1}{2}n-1} x^{m-1}$,)
 $4xy$

$4xym\dot{x} = 2x\dot{y} - 2xy.n\dot{x}$; therefore (x and y being equal,) $4mxy = 2nx^2 - 2nxy$, and $2my = nx - ny$, and $y = \frac{n}{2m+n} x$. Consequently, if $m=1$, and $n=2$, then $y = \frac{1}{3}x$, and $x = \frac{2}{3}a$; and, if $m=n$, then $y = \frac{1}{2}x$, and $x = \frac{1}{2}a$; as before.

Prob. 8.

20. *Let it be required to inscribe the largest rectangular Parallelogram in a given Semicircle.*

Common Way.

Put the Diameter AB (Fig. 3.) $= a$, AC $=$ DB $= x$, and CD $=$ EF $= a - 2x$; then, (by the Property of Circles) EC $=$ FD $= \sqrt{ax - x^2}$; and, (by Quest.) $\sqrt{a - 2x} \times \sqrt{ax - x^2} =$ a Maximum. Therefore, in Fluxions, $\frac{\frac{1}{2}a - x}{\sqrt{ax - x^2}} \times \frac{ax - 2xx}{\sqrt{ax - x^2}}$

$= 2 \cdot \frac{ax - x^2}{\sqrt{ax - x^2}} \dot{x} = 0$, that is, (multiplying by $\sqrt{ax - x^2}$;) $\frac{1}{2}a - x \times ax - 2xx = 2 \cdot \frac{ax - x^2}{\sqrt{ax - x^2}} \dot{x} = 0$, which (by Transposition and dividing by $\dot{x}$;) gives $\frac{1}{2}a^2 - 2ax + 2x^2 = 2ax - 2x^2$, or $\frac{1}{2}a^2 = 4ax - 4x^2$, or (dividing by 4,) $x^2 - ax = -\frac{1}{8}a^2$, which Quadratic solved, ($\frac{1}{2}a$ being more than x ;) gives $x = \frac{1}{2}a - \sqrt{\frac{1}{8}a^2} = a \times \frac{1}{2} - \sqrt{\frac{1}{8}}$.

New Way.

Put $AB=a$, $AC=DB=x$, and $CD=EF=a-2x$. — It is plain, that, as long as EF or CD increases faster in Proportion to its Length than DF or CE decreases in Proportion to its Length; or, *e contra*, as long as DF increases faster in Proportion than EF decreases, their Rectangle ED is increasing; but when either of those Dimensions begins to increase slower in Proportion to its Length than the other decreases, their Rectangle must consequently begin to decrease; that is, it will be a Maximum just where the two Dimensions are in direct Proportion to their Fluxions, or, where $EF : FD :: \text{Fluxion of } EF : \text{Fluxion of } FD$; that is, (because CE or $DF = \sqrt{ax-x^2}^{\frac{1}{2}}$,)

As $a-2x : \sqrt{ax-x^2}^{\frac{1}{2}} :: 2\dot{x} : \frac{\frac{1}{2}ax-x\dot{x}}{\sqrt{ax-x^2}^{\frac{1}{2}}}$, which

gives $2. \sqrt{ax-x^2}^{\frac{1}{2}}\dot{x} = \frac{\frac{1}{2}a^2\dot{x}-2ax\dot{x}+2x^2\dot{x}}{\sqrt{ax-x^2}^{\frac{1}{2}}}$, and

by multiplying by $\sqrt{ax-x^2}^{\frac{1}{2}}$, and dividing by $\dot{x}$, we shall have $2ax-2x^2 = \frac{1}{2}a^2-2ax+2x^2$, or $4ax-4x^2 = \frac{1}{2}a^2$, which gives, as before, $x =$

$\frac{1}{2}a - \sqrt{\frac{1}{8}a^2} = AC$ or BD : Consequently, GC

or $GD = \sqrt{\frac{1}{8}a^2}$, and CD or $EF = 2\sqrt{\frac{1}{8}a^2}$

$= \sqrt{\frac{1}{2}a^2}$, and CE or $DF = (\overline{AC \times CB})^{\frac{1}{2}} =$

$\sqrt{\frac{1}{8}a^2}$; and the Parallelogram or Rectangle
D required,

required, viz. $EF \times FD = \sqrt{\frac{1}{2}a^2} \times \sqrt{\frac{1}{2}a^2} = \sqrt{\frac{1}{4}a^2} = \frac{1}{2}a^2$; which is $\frac{1}{2}$ the Square inscribed in the Circle. So that the Point F will be just in the Middle of the quadrantal Arch BH,

N. B. In the above Solution, for the Fluxion of $a - 2x$ I have put $2\dot{x}$ and not $-2\dot{x}$, because all Fluxions considered in themselves are equally real and positive Quantities, whether they belong to increasing or decreasing Fluents; but more especially, as it appears from the Fig. that if the Fluxion of BD, or (which is the same) the Fluxion of AD, be $= \dot{x}$; then the Fluxion of CD must be $= 2\dot{x}$, as it flows both at C and D, and as the Velocity of the Point C is every where equal to the Velocity of the Point D.

Note. By the above Problem may be inscribed a Cylinder in a given Sphere, whose convex Surface shall be a *Maximum*.

SCHOLIUM.

21. FROM the New Way of investigating the foregoing Problems, it plainly appears, that, the common Method of making the Fluxion of a Maximum $= 0$, is right; the Fluxions of increasing and decreasing Quantities having different Signs prefixed. For Example; in the Solution of the first Prob. there are two Quantities x and y whose Sum is given $= a$: x is supposed to increase, and y to decrease (or *vice versa*,) and their Product, viz. xy , must be a Maximum: That is, by the new Method of

of Reasoning pursued in the foregoing Solutions, As $x : y :: \dot{x} : \dot{y}$, and therefore $x\dot{y} = y\dot{x}$; Consequently $x\dot{y} - y\dot{x} = 0$, which is the Fluxion of the Maximum xy , supposing one of the Quantities to increase and the other to decrease.

And, in the last Problem, if $2\sqrt{ax - x^2} \cdot \dot{x} = \frac{\frac{1}{2}a^2\dot{x} - 2ax\dot{x} + 2x^2\dot{x}}{ax - x^2|^{\frac{1}{2}}}$, then $\frac{\frac{1}{2}a^2\dot{x} - 2ax\dot{x} + 2x^2\dot{x}}{ax - x^2|^{\frac{1}{2}}}$

$= 2\sqrt{ax - x^2} \cdot \dot{x} = 0$, which is the Fluxion of the

Maximum $a - 2x \times \sqrt{ax - x^2}^{\frac{1}{2}}$. This then is a true Reason why the Fluxion of a Maximum is made $= 0$, when the Fluxions of increasing and decreasing Quantities are affected with contrary Signs, and when the Maximum is the Product of two flowing Quantities; viz. because each Quantity being to be multiplied into the Fluxion of the other, the Products must be equal; and consequently, one being subtracted from the other, the Difference must be $= 0$.

Note. All Products may be considered, each as the Product of two Quantities only; (as is very easy to conceive;) and with Regard to flowing Quantities, the Case must be the same; (e. g.) if any Number of Fluents be multiplied together, it is easy to consider the Product of the increasing Fluents as one Quantity, and the Product of the decreasing Fluents as another Quantity; which two Quantities are balanced, by being multiplied reciprocally into each other's Fluxion, as well as if they were simple Quantities; as may be partly seen in the Solution of the 4th, and more fully in some of the following Problems.

Prob. 9.

22. It is required to inscribe the largest Cylinder in a given Sphere: That is, in the Diameter AB, to find the Point G or H, where HG or EF multiplied into the Square of FG or EH shall be a Maximum; for, as EF or GH is supposed to be the Altitude of the Cylinder, so the Square of FG or EH is every-where as its Base. (Fig. 4.)

Common Way.

Put $AC=CB=a$, $CG=CH=x$; then $GB=AH=a-x$, $AG=a+x$, and $GF=\sqrt{aa-xx}$; and by the Quest. $2x \times \overline{aa-xx}$, i.e. $2a^2x-2x^3 = a$ Maximum; the Fluxion of which is $2a^2\dot{x}-6x^2\dot{x}=0$, consequently $a^2=3x^2$, and $x=\sqrt{\frac{a^2}{3}}=\frac{a}{\sqrt{3}}=CG$ required.

N. B. Here, as in several of the foregoing Problems, it happens, (as it were accidentally,) that one of the Fluents or flowing Quantities has the Sign + and the other the Sign — prefixed; by which Means, the common Method naturally takes Place; viz. making the Fluxion of the Maximum = 0; for either of the Quantities may be taken indifferently for the increasing, and the other for the decreasing Fluent.

New

New Way.

Put $AC = a$, and $CG = x$; then $2x \times \overline{a^2 - x^2} = a$ Maximum, as before; and let the Fluxion of $a^2 - x^2$ be $2x\dot{x}$, (and not $-2x\dot{x}$, as in the common Method,) all Fluxions, as such, being of a positive Nature; then, according to the new Method of Reasoning before laid down, As $2x : a^2 - x^2 :: 2\dot{x} : 2x\dot{x}$; $\therefore 4x^2\dot{x} = 2a^2\dot{x} - 2x^2\dot{x}$, and $6x^2\dot{x} = 2a^2\dot{x}$, which divided by $2\dot{x}$ gives $3x^2 = a^2$, $\therefore x = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}}$, as before.

Hence, $2x$, or the whole Altitude of the Cylinder, is $= 2\sqrt{\frac{a^2}{3}} = \sqrt{\frac{4a^2}{3}}$; that is, equal to the Square Root of $\frac{4}{3}$ of the Square of the Diameter; and consequently equal to the Side of a Cube inscribed in the same Sphere.

Prob. 10.

23. Let there be a Cone inscribed in a given Sphere, whose convex Surface shall be a Maximum. That is, let the Point C in the Diameter AB be found, where the Perpendicular CD being erected, the Product of AD into DC may be a Maximum; for DC, the Radius, is as the Circumference of the Base. (Fig. 2.)

Common

Common Way.

Put $AB=a$, and $AC=x$; then $CB=a-x$,
 $CD=\sqrt{ax-x^2}^{\frac{1}{2}}$, $AD=\sqrt{ax}^{\frac{1}{2}}$, and $\sqrt{ax}^{\frac{1}{2}} \times$
 $\sqrt{ax-x^2}^{\frac{1}{2}} = \sqrt{a^2x^2-ax^3}^{\frac{1}{2}} = a$ Maximum; and
 consequently $a^2x^2-ax^3 = a$ Maximum; for, if
 the Root be a Maximum, the Square (or in-
 deed any other Power,) must be a Maximum
 also; and, as a is a fixed Quantity, whatever
 is a Maximum when multiplied into that Quan-
 tity, must be a Maximum without that Multi-
 plication; and therefore $ax^2-x^3 = a$ Maxi-
 mum. Now, the Fluxion of this is $2ax\dot{x}-3x^2\dot{x}$
 $=0$; therefore (by dividing by $x\dot{x}$) $2a-3x$
 $=0$, and $3x=2a$, and $x=\frac{2}{3}a$.

New Way.

Put $AB=a$, and $AC=x$; then $ax^2-x^3 = a$
 Maximum, as before; *i. e.* $a-x \times x^2 = a$
 Maximum. Therefore, according to the new
 Method already explained, As $a-x : x^2 :: \dot{x}$
 $: 2x\dot{x}$; $\therefore x^2\dot{x} = 2ax\dot{x}-2x^2\dot{x}$, and $3x^2\dot{x} = 2ax\dot{x}$,
 and ($\div$ by $x\dot{x}$) $3x=2a$, and $x=\frac{2}{3}a$.

N. B. Here again, for the Fluxion of $a-x$,
 I have put $\dot{x}$, as denoting the positive Velocity
 of the flowing Quantity $a-x$, whether the said
 Quantity increases or decreases, even as $2x\dot{x}$ de-
 notes the positive Velocity of the Fluent x^2
 whether it increases or decreases.

Or,

Or,

24: If $AD \times DC$ (*Fig. 5.*) be a Maximum; then, by the new Method before laid down, the Fluxion of AD is to the Fluxion of DC as AD to DC . To illustrate this by a geometrical Construction, let An and nc be drawn indefinitely near to AD and DC ; drop the Perpendicular nr parallel to BA ; draw the Chord DB and the Tangent DT . Here then, Dr may represent the Fluxion or Decrement of DC , and qn the Fluxion or Increment of AD : If therefore Dr be to qn as DC to DA , then the Product of the two last is a Maximum; and this happens just as the Line BT is equal to AB ; for then, as $Dr : qn :: Dt : tn :: DB : BT :: DB : BA :: DC : DA$. Now, from the Line BT thus determined, may easily be found AD or BD ; for $AT \times BT = \overline{TD}^2 = \overline{TC}^2 + \overline{CD}^2$; that is, (putting $AB = a$, and $AC = x$,) $2a^2 = 4a^2 - 3ax$; therefore, by Transposition, $3ax = 2a^2$; $\therefore x = \frac{2}{3}a$; as before.

Note. The above Proportions suppose the Triangles $Dr t$ and $nq t$ in their nascent State, or, just beginning to exist by an infinitely small Opening of the Lines AD and An ; that is, the Angle DAn is supposed infinitely little; for then only it is that the said Triangles $Dr t$ and $nq t$ are similar, or that the Triangle Dtn is similar to the Triangle DBT . But, to demonstrate the Similarity, &c. of these Triangles; let the Line An be supposed to move towards AD till the

the said Lines are so near that $Dq =$

$\frac{1}{1000000000000} \times DA$; in this Case, Aq will be $= AD$, or (as we may say) infinitely near an Equality; consequently qn may be called the Fluxion or Increment of AD ; as rD is the Fluxion or Decrement of CD in the same infinitely small Portion of Time. Again, as in this Case the Angle DAq is infinitely small; so the Angle DqA must be right, or, which is the same in effect, infinitely near to a right Angle. Hence the Triangles $Dr t$ and $nq t$ are similar, because of the right Angles at r and q , and the vertical Angles at t : Consequently, As $Dr : qn :: Dt : tn$. Farther, by the above Supposition, the Points D and n will be infinitely near each other, and consequently the Tangent TD will approach infinitely near the Curve at n ; by which Means the Triangles Dtn and DBT become similar, or, which is in Effect the same, infinitely near to Similarity: So that, As $Dt : tn :: DB : BT$; and, as $BT = BA$, it will be, As $Dt : tn :: DB : BA$; but the Triangles DBA and CDA are similar, because of the right Angles at C and D and the common Angle at A : So that, As $DB : BA :: CD : DA$, *i. e.* As $Dt : tn :: CD : DA$; and $Dr : qn :: CD : DA$. Q. E. D.

Prob. 11.

25. In a given Ellipsis $ADBE$, it is required to inscribe the largest Triangle. (Fig. 6.)

Common

Common Way.

Let AFH be the Triangle required; and put AC or CB = a , CG = x ; Consequently AG = $a+x$, and GB = $a-x$: also put CD or CE

= b . Then, As AC $\times$ CB : $\overline{CD}^2 ::$ AG $\times$

GB : $\overline{GH}^2$; i. e. As $a^2 : b^2 :: a^2 - x^2 :$

$\frac{b^2 a^2 - b^2 x^2}{a^2}$ (= $\overline{GH}^2$.) Consequently $2 \times$

$\frac{b^2 a^2 - b^2 x^2}{a^2} = \text{HF}$, and $2 \cdot \frac{b^2 a^2 - b^2 x^2}{a^2} \times$

$a+x = \text{AG} \times \text{HF} = \text{the Maximum required;}$

the Fluxion of which is $\frac{b^2 a^2 - b^2 x^2}{a^2} - \frac{1}{2}$

$\times - 2 \frac{b^2 x \dot{x}}{a^2} + 2 \frac{b^2 a^2 - b^2 x^2}{a^2} \dot{x} = 0$; i. e.

$\frac{a+x}{a^2} \times - 2 \frac{b^2 x \dot{x}}{a^2} + 2 \frac{b^2 a^2 - b^2 x^2}{a^2} \dot{x} = 0$;

i. e. (by multiplying by $a^2 \cdot \frac{b^2 a^2 - b^2 x^2}{a^2}^{\frac{1}{2}}$) —

$2ab^2x\dot{x} - 2b^2x^2\dot{x} + 2b^2a^2\dot{x} - 2b^2x^2\dot{x} = 0$, or, —

$2ab^2x\dot{x} - 4b^2x^2\dot{x} + 2b^2a^2\dot{x} = 0$, therefore, (by dividing by $2b^2\dot{x}$), — $ax - 2x^2 + a^2 = 0$, or, $2x^2 +$

$ax = a^2$, or $x^2 + \frac{1}{2}ax = \frac{1}{2}a^2$. Consequently, $x^2 +$

$\frac{1}{2}ax + \frac{1}{16}a^2 = \frac{9}{16}a^2$; the Square Root of which

Equation is $x + \frac{1}{4}a = \frac{3}{4}a$; therefore, $x = \frac{1}{2}a$.

New Way.

According to the new Method before laid down, the Product of $AG \times GH$ will be a Maximum, when the Fluxions of those Quantities are in direct Proportion to the Quantities themselves: That is, let the right Line gb be drawn infinitely near and parallel to GH , and the Perpendicular bn parallel to gG ; then the Lines Hn and nb may stand for the Fluxions of HG and AG ; so that, GT (TH being a Tangent to the Point H ,) must be equal to GA ; for then $Hn : nb :: HG : GT :: HG : GA$. From hence the Point G may be easily found without a fluxional Operation; for, calling AC , a ; and CG , x ; we have this Proportion, viz. As $CG : CA :: CA : CT$, by the Property of the Curve, i. e. $x : a :: a : \frac{a^2}{x} = CT = a + 2x$. Consequently, $a^2 = ax + 2x^2$, as before.

Prob. 12.

26. Let it be required to inscribe the largest Cone, as AHF , in a given Spheroid, as $ADBE$. (Fig. 6.)

Common Way.

Put AC or $CB = a$, $CD = b$, $CG = x$, $AG = a + x$, and $GB = a - x$; then, by the Property of Ellipses, As $a^2 : b^2 :: a^2 - x^2 : \frac{a^2b^2 - b^2x^2}{a^2} =$

$= \overline{GH}^2$: Now $AG \times \overline{GH}^2 = \overline{a+x} \times \frac{a^2b^2 - b^2x^2}{a^2} = a \text{ Maximum, by the Quest.}$

therefore in Fluxions, $\frac{a^2b^2 - b^2x^2}{a^2} \dot{x} - \overline{a+x} \times$

$\frac{2b^2x}{a^2} \dot{x} = 0$; consequently $a^2b^2 - b^2x^2 = \overline{a+x}$

$\times 2b^2x$, and $a^2 - x^2 = \overline{a+x} \cdot 2x$, and $a^2 = 2ax +$

$3x^2$, and $\frac{a^2}{3} = x^2 + \frac{2}{3}ax$, and $\frac{4}{9}a^2 = x^2 + \frac{2}{3}ax +$

$\frac{5}{9}a^2$, and $\frac{2}{3}a = x + \frac{1}{3}a$, and $x = \frac{1}{3}a$.

New Way.

By the new Method before laid down, As $AG : \overline{GH}^2 :: \text{Fluxion of } AG : \text{Fluxion of } \overline{GH}^2$, (the Maximum being $AG \times \overline{GH}^2 = \overline{a+x} \times \frac{a^2b^2 - b^2x^2}{a^2}$, as before; where $AC=a$,

$CD=b$, and $CG=x$;) that is, as $a+x : \frac{a^2b^2 - b^2x^2}{a^2} :: \dot{x} : \frac{2b^2x}{a^2} \dot{x}$, (See *Art. 20. or*

23. N. B.) Consequently, $\frac{a^2b^2 - b^2x^2}{a^2} \dot{x} =$

$\overline{a+x} \times \frac{2b^2x}{a^2} \dot{x}$, and $a^2b^2 - b^2x^2 = \overline{a+x} \times 2b^2x$,

as before.

Prob. 13.

27. Let it be required to inscribe the largest Cylinder in a given parabolic Spindle. (Fig. 7.)

Common Way.

Put AC or $BC = a$, CE or $CG = x$, and $CD = b$; then, by the Property of the Parabola, $BC : \overline{CD}^2 :: BE : \overline{EF}^2$, i. e. As $a : b^2 :: a - x : \frac{ab^2 - b^2x}{a} = \overline{EF}^2$, which drawn into EG must be a Maximum, by the Quest. that is, $\frac{2ab^2x - 2b^2x^2}{a} = \text{a Maximum}$, which thrown into Fluxions is $\frac{2ab^2\dot{x} - 4b^2x\dot{x}}{a} = 0$; therefore, $2a - 4x = 0$, and $x = \frac{1}{2}a$.

New Way.

Put $AC = a$, $CD = b$, $CE = x$; then $\frac{2ab^2x - 2b^2x^2}{a} = \text{a Maximum}$, as before: And

therefore (see Art. 23.) $ax - x^2 = \overline{a - x} \times x = \text{a Maximum}$; and by the new Method of Reasoning, $a - x : x :: \text{Fluxion of } a - x : \text{Fluxion of } x$; that is, (Art. 20. Note) $a - x : x :: \dot{x} : \dot{x}$; Consequently $a - x = x$, and $x = \frac{1}{2}a$, as before.

Prob.

Prob. 14.

28. Let it be required to find the least Triangle AED that will circumscribe the given Rectangle HFKI. (Fig. 8.)

Common Way.

Put $AC=x$, $CB=a$, $CH=b$; then, $AC : CH :: AB : BD$, i. e. $x : b :: x+a :$

$\frac{bx+ab}{x} = BD$, which drawn into BA is

$$b + \frac{ab}{x} \times \frac{x+a}{x} = ab + bx + \frac{a^2b}{x} + ab =$$

$2ab + bx + \frac{a^2b}{x}$ a Minimum; the Fluxion

of which is $bx - \frac{a^2bx}{x^2} = 0$; therefore $bx^2 -$

$a^2bx = 0$, and $x^2 - a^2 = 0$, and $x^2 = a^2$, and $x = a$.

New Way.

Put $AC=x$, $CB=a$, $CH=b$; then $AC : CH :: AB : BD$, i. e. $x : b :: x+a :$

$b + \frac{ab}{x} = BD$; which drawn into BA is a Mini-

mum, by the Quest. that is, $b + \frac{ab}{x} \times \frac{x+a}{x}$

= a Minimum. Now, by the new Method of

Reasoning, $b + \frac{ab}{x} : a+x :: \frac{abx}{x^2} : x$, (con-

sidering

sidering the Fluxion of $\frac{ab}{x}$ as a positive Quantity,) therefore $b\dot{x} + \frac{ab\dot{x}}{x} = \frac{a^2b\dot{x} + abx\dot{x}}{x^2}$, which multiplied by x^2 gives $bx^2\dot{x} + abx\dot{x} = a^2b\dot{x} + abx\dot{x}$, therefore $bx^2\dot{x} = a^2b\dot{x}$, and $x^2 = a^2$, and $x = a$; as before.

SCHOLIUM.

29. HAVING proceeded thus far, I found, that the Method here proposed as a *new* one, has been observed, and followed by others. However, I have not seen the Reason and Foundation of this Method so clearly proposed by any Person as I have done it in the Beginning of this Section.

It is possible indeed, that this Method may have been used, not as it is immediately founded in the Nature of Fluxions (according to what was observed in the Beginning of this Section,) but only as it is a Consequence of the common Method of making the Fluxion of a Maximum or Minimum $= 0$: For, in that Case, the Fluxions of the increasing Quantities have the Sign $+$, and the Fluxions of the decreasing Quantities the Sign $-$ prefixed; (e.g.) if $ax - x^2 = a$ Maximum, then $a\dot{x} - 2x\dot{x} = 0$, (putting a for AB, and x for any Part of it, as AC, Fig. 2.) Now, it is plain, that the Fluxion of $a - x$ must be always equal to the Fluxion of x ; and consequently, when their Product $(ax - x^2)$ is a Maximum, the said Quantities themselves must be as their Fluxions, and

and therefore equal, (*i. e.*) $a - x = x$, because $ax - xx = xx$, which is the same as $ax - 2xx = 0$.

But, from this Way of considering the Fluxions of Maxima and Minima, is most easily deduced and demonstrated, the Method of making the Fluxion of a Maximum or Minimum $= 0$.

Prob. 15.

30. Let it be required to find a numerical Expression for the Quantity x , the Difference between the m th and n th Roots of which is a Maximum;

That is, to find the Value of x when $x^{\frac{1}{m}} - x^{\frac{1}{n}} =$ a Maximum.

Solution.

Here the Fluxion of $x^{\frac{1}{m}}$ is equal to the Fluxion of $x^{\frac{1}{n}}$; that is, $\frac{1}{m} x^{\frac{1}{m}-1} \dot{x} = \frac{1}{n} x^{\frac{1}{n}-1} \dot{x}$;

therefore $\frac{n}{m} x^{\frac{1-n}{m}} = x^{\frac{1-n}{n}}$, and consequently

$\frac{n}{m} = x^{\frac{m-n}{mn}}$, and $x = \frac{n}{m}^{\frac{mn}{m-n}}$; which, supposing $m=3$, and $n=2$, is $x = \frac{64}{729}$. Q. E. I.

Note. For a Proof of the above Assertion, viz. that the Fluxion of $x^{\frac{1}{m}}$ is $=$ the Fluxion of $x^{\frac{1}{n}}$, when their Difference is a Maximum;

Let it be observed, that, from the Nature of the Question, x must necessarily be a fractional Quantity, and consequently m must be greater than n in order to make x^m greater than x^n , or nearer to *Unity* (which is the infinite Root of all Quantities both whole-number and fractional.) From which Lemma it will also follow, that, the nearer any Quantity is to *Unity*, the less will be the Differences of its gradually descending Roots; since by an infinite Progression they can proceed no farther than *Unity*; And therefore, if the Quantity in Question were more than *Unity*, it must be removed at an infinite Distance from it (*i. e.* it must be infinitely great) in order to produce the Maximum required. But, if it be removed on the other Side, and supposed, as it necessarily must be, less than *Unity*; here it comes to be considered, that the farther it is removed from *Unity*, the less the Quantity itself will be: And, that, if the Quantity itself be supposed infinitely distant from *Unity*, (*i. e.*) infinitely small, the Difference in Question will be infinitely small likewise; which will also be the Case if the Quantity be supposed infinitely near to *Unity*. So that, from *Unity* downwards, the said Difference must increase for some Time, and after that, continually decrease: Now, if the Causes of this Increase and Decrease be compared, it will discover their separating Point. The Cause of the Increase is, the Quantity's gradually departing from *Unity*; and the Cause of the Decrease, is, the continual Lessening of the Quantity. These Causes both operate continually; and as long as the former operates more strongly than the latter,

latter, the Difference in Question must increase; but when the latter begins to operate more strongly than the former, just then is the required Maximum produced. — Or, to express it more accurately and plainly; as long as the Distance of the greater Root of the Quantity, from *Unity*, flows swifter than the Decrease of the lesser Root of the said Quantity, the aforesaid Difference increases; but, as soon as the latter begins to flow swifter than the former, the said Difference begins to decrease; that is,

the Fluxion of $x^{\frac{1}{n}}$ is $\equiv$ the Fluxion of $x^{\frac{1}{n}}$; for the Fluxion of $1 - x^{\frac{1}{n}}$ (the Distance from *Unity*) $\equiv$ the Fluxion of $x^{\frac{1}{n}}$, and the Fluxion of $x^{\frac{1}{n}}$ is the Fluxion or Velocity of its Decrease.
Q. E. D.

Prob. 16.

31. Given two Circles in Magnitude and such Position as not to touch each other: Required to draw the shortest Line possible between their Peripheries that shall make a given Angle with the Line joining their Centers: That is, given the Radii of the two Circles AD and BF, and the Distance of their Peripheries DF; Required the Minimum EG intersecting the Line AB in a given Angle as DAc or FBL. (Fig. 9.)

F

Solution

Solution.

Complete the rectangular Parallelogram AIBH, and suppose the Line AI to move on in a perpendicular Direction towards HB, or so as always to be parallel to its first Situation; then the Part *eg* intercepted between the two Peripheries, will continually shorten at the End *g*; and lengthen at the End *e*, till it gets clear of the Periphery that bounds it at that End: But, it will decrease with a retarded or lessening Velocity at the End *g*, and increase with an accelerated or increasing Velocity at the End *e*. And as the Velocity at *g*, is infinitely greater than that at *e*, at the Beginning of the Motion; so, where those two Velocities or Fluxions become equal, there is the Minimum required; *viz.* at EG, where the Tangents of the two Circles are parallel to each other: For, as before it came thither, it shortened at the End *g* faster than it lengthened at *e*; so, after it passes that Place, it will lengthen faster at the End *E* than it will shorten at *G*: *Ergo, &c.* ——— Now, as the Tangents of the two Circles at *E* and *G* are parallel (from the Nature of the Quest.) so, consequently, the Arches DE and FG must be similar, as also the Triangles AEC and BGC; from which it will follow, that, the Point of Intersection *C*, divides both DF and EG, in the Proportion of the Radii AD and BF. So that, if $AD=a$, $BF=b$, and $DF=c$; then $DC = \frac{ac}{a+b}$; for $AD+BF : AD :: DF : DC$. Thus, in the Triangle AEC, there are given
AE

AE and AC and the Angle ACE, to find CE; a common Case in plain Trigonometry; and CG is a fourth Proportional to AE, EC, and BG.

Prob. 17.

32. Let GC be perpendicular to CB, $AC = a$, and $AB = b$; and let the Line AB move, viz. its End B directly towards C, and its End A round the Curve AEG, so as that the said Line shall continually pass through the Point A of the Line GA. Required the Perpendicular EF, the greatest Distance of the said Curve from the Line GA. (Fig. 10.)

Solution.

It is evident, that, the Curve AE continually recedes from the Line AG till its Direction or (which is the same) its Tangent becomes parallel to AG, as EH at the Point E; after which, it continually draws nearer to the said right Line AG again. Let the pricked Line ed be supposed indefinitely near to ED, and to pass through the Point A; and draw the indefinitely small Arches db and Ea on A as a Center. From hence it will easily appear, that, the small fluxional right-angled Triangles Dbd and Eae are, in their nascent State, respectively similar to the right-angled Triangle DHE, i. e. to DCA or EFA. It is also very evident, that, the Fluxion (or Swiftneſs of Increase) of the Line AD, i. e. ae , is equal to Db . It is farther evident, that, the Fluxions Ea and db are

directly as the Parts AE and AD, viz. as the Radii of the Circles of which they are similar Arches. — Now, put $AD=r$; conse-

quently $AE=b-r$, and $CD=\sqrt{r^2-a^2}$. A-

gain, put bD the Fluxion of $AD=r=ae$ the Fluxion of AE ; and let $bd=j$; consequently

$Ea=\frac{b-r \cdot j}{r}$. Then, according to the above

Observations, as $Db : bd :: Ea : ae = Db$,

i. e. $r : j :: \frac{b-r \cdot j}{r} : r$; consequently $r^2 =$

$\frac{b-r \cdot j^2}{r}$, and $rr^2 = b-r \cdot j^2$; and therefore,

as $b-r : r :: r^2 : j^2$, i. e. As $b-r : r ::$

$Db^2 : bd^2$ and so DC^2 to CA^2 , i. e. fo

r^2-a^2 to a^2 ; i. e. $b-r : r :: r^2-a^2 : a^2$,

$\therefore a^2b-a^2r=r^2-a^2r$, and $a^2b=r^3$; so that,

$r=\sqrt[3]{a^2b}$: from whence may be easily found

EF; for, as $AD : DC :: AE : EF$, i. e.

As $r : \sqrt{r^2-a^2} :: b-r : \frac{b-r \cdot \sqrt{r^2-a^2}}{r} =$

EF.

Note. The Equation of the Curve AEG may

be found thus. Put $AB=ED=CG=a$, AC

$=b$, Absciss $AF=x$, Ordinate $FE=y$; then

$AE=\sqrt{x^2+y^2}$. Now $AE : AF :: AD :$

AC , and therefore, $AE : AF :: DE : CF$,

i. e. $\sqrt{x^2+y^2} : x :: a : b+x$, $\therefore \frac{ax}{b+x} =$

✓

$\sqrt{x^2 + y^2}$, which is the Equation required. Hence the following geometrical Construction, viz. Make $CI = AB$, draw AK equal and parallel to CI , and from I to any Point F in the Axis, draw IF ; with the Radius AN (N being the intersecting Point of the Lines AK and FI), describe the circular Arch NE , and from the Point F draw the Perpendicular FE ; then will the intersecting Point E of the said Arch and Perpendicular be in the Curve in Question. For, draw FL equal and parallel to CI ; and IKL , ONM , equal and parallel to CF ; then $CFMO = AFLK = ax$; which divided by CF or $b + x$, quotes FM or $AN = \frac{ax}{b+x} =$

$\sqrt{x^2 + y^2} = AE$; therefore, &c.

33. *Note.* If the Lines EF and AC be given, and a *Maximum* of AB or ED be required; it is evident that the same Fig. will do; And that, as $EF : CD :: EA : AD$; but by the above Solution, $EA : AD :: \overline{DC}^2 : \overline{CA}^2$; therefore $EF : CD :: \overline{DC}^2 : \overline{CA}^2$; that is, calling EF , c ; CD , x ; and AC , a ; $c : x :: x^2 : a^2$; Consequently $x^3 = ca^2$, and $x = \sqrt[3]{ca^2}$; from whence DA , and consequently DE , may be easily found.

Prob. 18.

34. *Given the Distances AB and AC from the Perpendicular AD : Required the Point D where the Angle BDC is a Maximum, (Fig. FI.)*

Solution.

Solution.

The Fluxions of the Angles DBC and DCB must be equal; for, as soon as the Angle B gains as fast as the Angle C loses, the Angle D can gain Nothing more. — Draw the fluxional Radii Cd and Bd ; and the fluxional Arches dr and dt , on the Centers C and B. Then, for the above Reason, as $DC : DB :: dr : dt$, and by sim. Triangles, as $AC : CD :: dr : dD$; and as $AB : BD :: dt : dD$; $\therefore$ (calling

dD , 1) $\frac{AC}{CD} = dr$, and $\frac{AB}{BD} = dt$. Conse-

quently, as $DC : DB :: \frac{AC}{DC} : \frac{AB}{DB}$; $\therefore$

$\frac{AB \times DC}{DB} = \frac{AC \times DB}{DC}$, and $AB \times \overline{DC}^2 = AC \times$

$\overline{DB}^2$; $\therefore AB : AC :: \overline{DB}^2 : \overline{DC}^2$. So that the Lines DB and DC are in a subduplicate Ratio of the Lines AB and AC. Wherefore, call AC, a ; AB, b ; and AD, p ; and then

$\sqrt{a^2 + p^2} : \sqrt{b^2 + p^2} :: \sqrt{a} : \sqrt{b}$; or, $a^2 + p^2 : b^2 + p^2 :: a : b$; $\therefore ab^2 + ap^2 = a^2b + p^2b$,

and $ab^2 - a^2b = bp^2 - ap^2$, and $\frac{ab^2 - a^2b}{b - a} = p^2$, and

$p = \sqrt{\frac{ab^2 - a^2b}{b - a}} = \sqrt{ab}$. Thus the Distance

AD is a mean Proportional between the Distances AB and AC.

Corollaries.

Corollaries.

1. The Triangles B A D and D A C, are similar.

2. The Fluxions of the Lines CD and BD at the Point required, are in the reciprocal Ratio of the Lines themselves; for, from sim. Triangles, as Dt : $Dd (=1)$:: DA : DB; and, as Dr : $Dd (=1)$:: DA : DC. Consequently $Dt = \frac{DA}{DB}$, and $Dr = \frac{DA}{DC}$; then

$Dt \times DB = DA = Dr \times DC$; So that, as Dt : Dr :: DC : DB.

Prob. 19.

35. Given the Distances AB and AC, and the Position of the Line AD; Required the Point D where the Angle BDC is a Maximum. (Fig. 12. and 13.)

Solution.

It is evident that this cannot happen as long as the Fluxion of the Angle ACD is greater than that of the Angle ABD; because all that while the Angle BDC must be increasing; and therefore the said Angle BDC must arrive at its Maximum just as the said Fluxions come to an Equality; *i. e.* when the little fluxional Arches dr and dt are in the direct Proportion of their Radii dC and dB ; for after that, the Fluxion of the Angle ABD will be greater than that of the Angle ACD, and consequently the Angle

BDC

BDC must decrease. — From the Points C and B, let the Perpendiculars CG and BH be dropt on the Line DA produced, which Perpendiculars are plainly proportional to the two given Distances AC and AB. So that, if $AC = a$, and $AB = b$, then CG may be $= ra$, and $BH = rb$; and, calling AD , p ; and DD , r ; it will be, as $DC : DB :: dr : dt$; and by sim. Triangles, $dr : Dd :: GC : CD$; *i. e.* as $dr : 1 :: ra : CD$; therefore (putting $CD = x$) $dr = \frac{ra}{x}$. Again, as $dt : dD :: BH : BD$, *i. e.* as $dt : 1 :: rb : BD$; therefore (putting $BD = y$) $dt = \frac{rb}{y}$. Consequently, (as $CD : BD :: dr : dt$, *i. e.*) as $x : y :: \frac{ra}{x} : \frac{rb}{y}$, and $\frac{ra}{x} = \frac{rb}{y}$, and $ray^2 = rbx^2$, and $ra : rb :: x^2 : y^2$. So that the Lines DC and DB are in the subduplicate Ratio of the Lines GC and HB, and consequently of AC and AB. — Now, to find the Distance AD, Say, as $\overline{CD}^2 : \overline{BD}^2 :: a : b$, *i. e.* As $\overline{GC}^2 + \overline{GD}^2 : \overline{HB}^2 + \overline{HD}^2 :: a : b$; or, (Fig. 12.) as $a^2 + p^2 + 2p\sqrt{a^2 - r^2a^2} : b^2 + p^2 + 2p\sqrt{b^2 - r^2b^2} :: a : b$; or (Fig. 13.) as $a^2 + p^2 - 2p\sqrt{a^2 - r^2a^2} : b^2 + p^2 - 2p\sqrt{b^2 - r^2b^2} :: a : b$. Whence, $a^2b + p^2b = ab^2 + ap^2$, and $p^2b - p^2a = ab^2 - a^2b$, and $p^2 = ab$, and $p = \sqrt{ab} = AD$. So that the Distance AD is a mean

mean Proportional between the Distances AB and AC; and the Triangles BAD and DAC are similar.

N. B. In this Solution, the Point *d* is supposed to be infinitely near to the required Point D; and the little Arches *dt* and *dr*, described with the Lines *Bd* and *Cd* as Radii, to be right Lines perpendicular to BD and CD.

Or thus, by the Editor:

36. It is evident, that, as long as the Angle DBC (*Fig. 14. 15. and 16.*) increases slower than the Angle DCB decreases, the Angle CDB will be increasing; and that, when the Angle DCB decreases slower than the Angle DBC increases, the Angle CDB will be decreasing: Therefore, when the said Angle CDB is a Maximum, or the greatest possible, the Fluxions of the Angles at C and B will be equal; *i. e.* if we suppose the Point *d* indefinitely near to D, $\angle DCd = \angle DBd$; and therefore, if with BD and CD, as Radii, the Arches *Dm* and *Dn* be described, then $Dm : Dn :: BD : CD$. Draw the Circle DGA, whose Diameter is AD; and in *Fig. 16.* produce BD to G, and DC to I. Draw likewise the right Lines GA, AI, and IG. Then, in *Fig. 14. and 15.* $\angle GAI = \angle GDI$ or BDC; and in *Fig. 16.* $\angle GAI + \angle GDI = \angle GDI + \angle BDI$, and therefore in this *Fig.* also $\angle GAI = \angle BDI$ or BDC. Now, if we suppose *Dm* and *Dn* to be little right Lines perpendicular to *Bm* and *Cn* respectively, then will the Triangles *dDm* and *DAG* be similar, as will likewise the Triangles *dDn* and *DAI*. Therefore, $\angle GDI = \angle BDI$ or BDC.

angles dDn and DAI ; and therefore, $dD : DA :: Dm : AG$, and $dD : DA :: Dn : AI$; $\therefore Dm : Dn :: AG : AI$. Hence we have $BD : CD :: AG : AI$; and consequently the Triangles BDC and GAI are alike, and $\angle AGI = \angle DBC$. But, $\angle AGI = \angle ADI$ or ADC ; therefore, $\angle DBC = \angle ADC$, and the Triangles CAD and DAB are similar; wherefore $CA : AD :: DA : AB$, and $AD =$

$\sqrt{AC \times AB}$. Hence the following Construction, viz. Produce BA to K , making $AK = AC$; describe the Semicircle KB ; and, in Fig. 14. and 16. draw AQ perpendicular to the Diameter BK ; also, make $AD = AQ$: Then will D be the Point required.

Prob. 20.

37. Given the Radius AE , and the Distances EC and EB on the same Radius produced. Required the Point D in the Periphery where the Angle CDB is a Maximum. (Fig. 17.)

Solution.

This must be where the Fluxions of the Angles ECD and EBD are equal; for as long as the Fluxion of the $\angle ECD$ is greater than that of the $\angle EBD$, the $\angle CDB$ must gain; and as soon as the Fluxion of the $\angle EBD$ exceeds that of ECD , then the $\angle CDB$ must decrease; *ergo*, &c. — From the Point d infinitely near to D , draw dC and dB , and the fluxional Arches dt and dr on the Centers B and C ; lastly draw the Tangent DH , and drop the Perpendiculars.

diculars CG and BH. Put $AC=a$, $CB=b$, $AE=c$, and $Ca=x$. ——— Now, if it were demanded in what Point of the right Line aD the Angle subtended by CB would be a Maximum; it is evident, that, this Point would coincide with the Point D in the Circle; because, in that Point only, are the Fluxions of the Angles C and B equal. Therefore, by the preceding Prob. aD is a mean Proportional between aC and aB , or, between x and $b+x$;

$$\text{that is, } aD = \sqrt{bx + x^2} = \sqrt{Aa^2 - AD^2} \\ = \sqrt{a^2 - 2ax + x^2 - c^2}. \text{ So that } bx = a^2 - 2ax - c^2, \text{ and } 2ax + bx = a^2 - c^2, \text{ and } x = \frac{a^2 - c^2}{2a + b} =$$

$$Ca: \text{ Consequently } Aa = a - \frac{a^2 - c^2}{2a + b} = \frac{a^2 + ab + c^2}{2a + b}.$$

Here then, we have the Secant of the Arch in Question; from whence the Arch itself, or the Point D, may be easily found.

C H A P. H.

MISCELLANEOUS QUESTIONS WITH
THEIR SOLUTIONS.

I.

IF A puts out a Sum of Money to B, in Order to receive double the natural Interest of it yearly, during his Life; and at his Death, the original Sum to be B's: *Quare*, how long ought A to live, that B may be neither a Gainer nor Loser?

Solution, by the Editor.

Let p = the Sum B receives, r = the Amount of 1 $l.$ with its Interest for 1 Year, and t = the Number of Years A ought to live. Then,

At the End of 1 Year, after
B has paid A 1 Year's Annuity, the Sum in B's Hands
will be $\left. \begin{array}{l} = pr - : 2p.r - 1 \\ = 2p - pr. \end{array} \right\}$

At the End of 2 Years, $= \overline{2p - pr.r} - : \overline{2p.r - 1}$
 $= 2p - pr^2.$

At the End of 3 Years, $= \overline{2p - pr^2.r} - : \overline{2p.r - 1}$
 $= 2p - pr^3.$

At the End of 4 Years, $= \overline{2p - pr^3.r} - : \overline{2p.r - 1}$
 $= 2p - pr^4.$

And,

And, therefore, at the End of t Years it will be $= 2p - pr^t$, which by the Quest. must be $= 0$; therefore $2p = pr^t$, and $2 = r^t$. Hence, by the Nature of Logarithms, $t \times \text{Log. of } r$ is $= \text{Log. of } 2$: Consequently $t = \frac{\text{Log. } 2}{\text{Log. } r}$ (supposing $r = 1.05$, or the Interest to be 5 per Cent.) 14 Years and 75 Days. Q. E. I.

Note. If A , instead of *double*, were to receive a Times the natural or simple Interest; then, by the above Method of Reasoning, we

$$\text{shall have } t = \frac{\text{Log. } \frac{a}{a-1}}{\text{Log. } r}$$

II.*

There is a Wall erected perpendicular on a Plain; and upon the Ground, against the Foot of the Wall, lies a Cask, whose external Bung Diameter is 4 Feet; over which a Ladder 13 Feet long is to be erected: Quere, how high may it be erected? — That is, in other Words, given the Hypothenufe AB of the right-angled Triangle ACB , and the Diameter of the inscribed Circle DEF ; to find the Perpendicular CB . (Fig. 18.)

Solution.

Let p = the Perpendicular required = CB , and b = the Base = AC ; and put h = the Hypothenufe

* Written in the Year 1745.

pothenuse $= AB = 13$, and $r =$ the Radius of the inscribed Circle $= \odot F = \odot E = \odot D = 2$. Now, (drawing the right Lines $A\odot$ and $\odot B$;) the $\triangle AB\odot = \triangle s E\odot B + D\odot A$: Consequently $\triangle ABC =$ twice $\triangle A\odot B + \square D\odot EC$; i. e. $br + rr = \frac{1}{2}bp$; and therefore $4br + 4rr = 2bp$: But, by Pythagoric Theorem, $bb = pp + bb$; which Equation added to the last, gives $bb + 4br + 4rr = pp + 2bp + bb$, the Square Root of which is $b + 2r = p + b$; the Truth of which curious Theorem appears geometrically from a bare Inspection of the Figure only. By subtracting $2bp$ from $pp + bb$, we have $bb - 4br - 4rr = pp - 2bp + bb$; the Square Root of which is $\sqrt{bb - 4br - 4rr} = p - b$. Consequently

$b + 2r + \sqrt{bb - 4br - 4rr} = 2p$. Hence therefore, $p = \frac{b + 2r + \sqrt{bb - 4br - 4rr}}{2} = 12$.

Q. E. I.

III.

An infinite geometrical Series, decreasing from Any-thing to Nothing, as $a + b + \frac{b^2}{a} + \frac{b^3}{a^2} + \&c.$ is equal to the Square of the 1st Term divided by the Difference between the 1st and 2nd $= \frac{aa}{a-b}$.

Required the geometrical Demonstration?

Demonstration.

Demonstration.

In the indefinite right Line ABC &c. (Fig. 19.) let AB represent the 1st Term, and BC the 2nd in the Series; upon which erect the Squares ABIH and BCLK; then through the Points H and K, let a right Line be produced, and it will cut the indefinite right Line ABC &c. in the Point P. Now, the Line AP will truly represent the Sum or Aggregate of the infinite Series in Quest. For the Squares may be supposed to be continued on *in infinitum* as to Number, and their Sides must inevitably bear the same Proportion with the Terms in the Series: And that the Squares cannot go beyond the Point P, is self-evident, since the Side of every new Square to be erected will be to the Residue of the Line AP as HA to AP, or KB to BP, or MC to CP, &c. Now, to find the Length of AP, or the Sum of the infinite Series, it is plain that the Triangles KIH and HAP are similar, and consequently, as KI to HI so HA to AP; that is, $a - b : a :: a :$

$$\frac{aa}{a-b} = AP. \quad Q. E. D.$$

N. B. The same Conclusion will follow from any other of the Triangles above the slope Line, e. g. MLK; for ML : LK :: HA : AP; i. e. $b -$

$$\frac{b^2}{a} : b :: a : \frac{ab}{b - \frac{b^2}{a}} = \frac{aab}{ab - b^2} = \frac{aa}{a - b}, \text{ as be-}$$

fore. — Or, ML : LK :: KB : BP; i. e. $b -$

$$b - \frac{b^2}{a} : b :: b : \frac{b^2}{b - \frac{b^2}{a}} = \frac{ab^2}{ab - b^2} = \frac{ab}{a - b}$$

BP, to which add AB = a , and then $\frac{aa}{a - b} =$
AP.

Corollary.

When the third Term is $= a - b$; that is, when the Terms in the above Series are as $a : b :: b : a - b$, or the common Ratio is that of a Line cut in mean and extreme Proportion;

then $\frac{aa}{a - b}$, the Sum of the said infinite Series,

will be $= \frac{a^2}{b^2}$, or $2a + b$. For, then $a - b =$

$\frac{b^2}{a}$, and consequently, $\frac{a^2}{a - b} = \frac{a^3}{b^2}$: Or, since a^2

$= ab + b^2$; i. e. $a^2 = ab + b^2$; therefore $\frac{a^2}{a - b} =$

$\frac{ab + b^2}{a - b} =$ (by substituting $\frac{b^2}{a}$ for $a - b$,)

$\frac{a^2b + ab^2}{b^2} = \frac{a^2}{b} + a =$ (by writing $ab + b^2$ for

a^2 ,) $\frac{ab + b^2}{b} + a = 2a + b$. (See the follow-

ing *Quest.* where this same Truth is demonstrated geometrically.)

IV.

An infinite geometrical Series, whose common Ratio is that of a Line cut in mean and extreme Proportion, is equal to twice the first Term added to the second. Required a geometrical Demonstration? (See the preceding Corollary.)

Demonstration.

Let the Sides AB and DC of the regular Pentagon ABCDE in Fig. 20. be produced till they intersect in M; then the Line AM will represent such an infinite Series as is supposed in Quest. as the Fig. will intuitively demonstrate. Now, if AB be the 1st Term, BG the 2nd, GI the 3d, &c. *in infinitum*; then, AM will be equal to $2AB + BG$; for $MB = BD$ (the Angles at M and D in the Triangle MBD being each $= \frac{1}{3}$ th of 2 right Angles,) and $BD = AC = AG$. So that $MB = AG$, and consequently $MA = 2AB + BG$. Q. E. D.

V.*

The Sum of the three Perpendiculars let drop from any Point in an equilateral Triangle to the three Sides, as $ED + EH + EK$, or $FG + FH + FM$, is always equal to CD the Perpendicular of the Triangle. Quære the Demonstration. (Fig. 21.)

H

Demonstration.

* Written in the Year 1747.

Demonstration.

1°. If the Point be taken in the Perpendicular, as at E, then $EH = EK = \frac{1}{2}EC$; for, from sim. Δ s, $CB : BD :: CE : EH :: 1 : \frac{1}{2}$; and therefore $EH + EK = EC$; and consequently $EH + EK + ED = CE + ED = CD$.

2°. If the Point be taken out of the Perpendicular, as at F; then produce the Perpendicular HF till it intersects the great Perpendicular CD, as at E: this will form the little equilateral Triangle EFL; in which let drop the Perpendiculars EN and FI. Then, it is evident, that, the Sum of the Perpendiculars FG, FH, FM, is equal to the Sum of the Perpendiculars EK, EH, ED; for, as FH falls short of EH by the Quantity EF, which is a Side of the little equilateral Triangle; so the other two, FM and FG, just make up that Deficiency; for $FM = ED + EI$, and $FG = EK + NF$: Now, the Lines EI and NF are each of them $\frac{1}{2}$ the Side of that little equilateral Triangle EFL. *Ergo*, &c. Q. E. D.

VI.

If any Triangle ABC circumscribe a Circle whose Center is E, and be inscribed in another Circle whose Center is F: then, if the Chord AG be drawn through the Center E of the inscribed Circle, and the Radius of the circumscribing Circle be put $= r$; I say EF, the Distance of the said Centers, will

will be $= \sqrt{r^2 - AE \times EG}$. *Quare the Demonstration?* (Fig. 22.)

Demonstration.

Through E draw the Diameter HI: Then, by 35. E. 3. $AE \times EG = HE \times EI =$
 $\overline{r - FE} \times \overline{r + FE} = r^2 - \overline{EF}^2$; $\therefore \overline{EF}^2$
 $= r^2 - AE \times EG$, and consequently $EF =$
 $\sqrt{r^2 - AE \times EG}$. Q. E. D.

Note. If the Sides of the Triangle be given; and from thence its Angles, as also EK and FG, (the Radii of the Circles,) and the Line AE, be found; then the Chord AG, and consequently EG, may be easily determined; for, $\angle AFC = 2 \angle ABC$, and $\angle CFG = 2 \angle CAG$; and therefore $\angle AFG = 2 \angle ABC + \angle CAB$.

VII.

On the right Line DC, let the Perpendicular CE be erected; and let the Line AB be equal to and coincident with DC; in which Case, F, the bisecting Point of AB, will coincide with G the bisecting Point of DC: Then, let the End B of the said Line AB move along the Line CE, whilst A the other End of it moves on from D to C. *Quare the Nature of the Curve which F the said bisecting Point of the Line AB will describe by Means of the Motion aforesaid?* (Fig. 23.)

H 2

Solution.

Solution.

Put $AB = DC = a$; then $AF = DG = GC = \frac{1}{2}a$; also, put $GH = x$, and $FH = y$ perpendicular to DC . Now, as the Direction of the Motion of F , the middle Point, is everywhere compounded of the two Directions of the two Ends A and B in their Motions: And, as the Velocities of the two component Motions of F are just Half of the Velocities of A and B respectively; (*i. e.* as the Velocity of the Point F , in the Direction of HF or CE , is = Half the Velocity of the Point B in the same Direction; and in the Direction of DC = Half the Velocity of the Point A ;) so the Line GH will be every-where = Half DA : Consequently $DA = 2x$, and $AC = a - 2x$; and as $AH = \frac{1}{2}AC$, (for $AB : AF :: AC : AH$), therefore $AH = \frac{1}{2}a - x$. Now, $\overline{AH}^2 + \overline{HF}^2 = \overline{AF}^2$, that is, $\frac{1}{4}a^2 - ax + x^2 + y^2 = \frac{1}{4}a^2$; and therefore $x^2 + y^2 = ax$, and $y^2 = ax - x^2 = a - x \times x$; which is the Equation of a Circle. Consequently the Curve GF is Part of a circular Circumference. Q. E. I.

Or,

To demonstrate the same *geometrically*. — Draw the right Line FC ; then, as AH appears from sim. Triangles to be $= \frac{1}{2}AC = HC$; so FC must be $= FA = DG = CG$. So that the Point F is every-where equidistant from C ; which

which Point C therefore, is the Center of a Circle described by the Point F. Q. E. D.

VIII.

In the preceding Problem; Quære the Nature of the Curve described by any (or every) other Point of the Line AB? (Fig. 24.)

Solution.

Put $AF=a$, and $FB=b$: Consequently $DG=CI=a$, and $GC=b$. Let $GH=x$, and $HF=y$. Then, by the foregoing Rule of Velocities, as $FB : AB :: GH : AD$, that is,

$$b : a+b :: x : \frac{ax+bx}{b} = \frac{ax}{b} + x = AD:$$

Conseq. $AG = a - \frac{ax}{b} - x$, and $AH = a -$

$\frac{ax}{b}$. Now, $\overline{AH}^2 + \overline{FH}^2 = \overline{AF}^2$, i. e. a^2

$$- \frac{2a^2x}{b} + \frac{a^2x^2}{b^2} + y^2 = a^2; \text{ therefore } y^2 =$$

$$\frac{2a^2x}{b} - \frac{a^2x^2}{b^2} = \frac{2a^2bx - a^2x^2}{b^2} = \frac{a^2}{b^2} \times \overline{2bx - x^2},$$

$$\text{and } \frac{y^2}{a^2} = \frac{2bx - x^2}{b^2}; \text{ wherefore, as } a^2 : b^2 :: y^2$$

$$: (2bx - x^2) = \overline{2b - x} \times x, \text{ or, } \overline{CI}^2 : \overline{CG}^2$$

$:: \overline{FH}^2 : \overline{2GC - GH} \times GH$; which being the Property of an Ellipsis, whose Semidiameters are CI and CG; therefore, the Curve in Question is the common *Ellipsis*. Q. E. I.

IX.

Quere the Nature of the Curve CDF, which is every-where equidistant from the Circumference of the Circle BDK and its Radius AD produced; that is, $BC=CA$, $fk=fe$, $FK=FE$? (Fig. 25.)

Solution.

Put Radius $AD=2AC=a$, Absciss CH or $Cb=x$, Ordinate HF or $bf=y$; then $AH=EF=KF=x-\frac{1}{2}a$, and $AF=x+\frac{1}{2}a$; or $Ab=\frac{1}{2}a-x$, and $Af=\frac{1}{2}a+x$. By 47. E. 1.

$\overline{FA}^2 = \overline{AH}^2 + \overline{HF}^2$, or $\overline{fA}^2 = \overline{Ab}^2 + \overline{bf}^2$; that is, $x^2 + ax + \frac{1}{4}a^2 = x^2 - ax + \frac{1}{4}a^2 + y^2$, or, (by Transposition,) $2ax = y^2$; which is the fundamental Property of the common *Parabola*; and therefore the Curve in Question is a parabolic Curve, whose Vertex is C, Focus A, and Semiparameter the Radius AD. Q. E. I.

Note. As FE is always equal to FK ; so the Curve must always bisect the Angle formed by those two right Lines in the Point F; for, if it leaned, or inclined, to one of the said Lines more than to the other, as it passed through the said Point; then, at the smallest conceivable Distance from that Point, it would be nearer to the Circle than to its Radius produced, or nearer to the latter than to the former, contrary to the Generation: The Tangent therefore, in the Point F, must bisect the $\angle AFE$, and make $\angle AFG$

$\angle AFG = \angle EFG$, (GF being a Tangent to the Curve;) but, EF being parallel to AG, $\angle EFG = \angle FGA$; consequently, $\angle AFG = \angle AGF$, and $AF = AG$. Again, since $AH = EF = KF$, therefore $AF = BH = AG$; and, because $BH = AG$, and $BC = CA$, therefore $CH = CG$. The same Reasoning may be easily applied to any Point, as f , within the Circle. Hence it follows, that,

If the right Line AL be drawn perpendicular to the Tangent GF; then will the said Line bisect both the said Tangent and the Angle GAF.

If on the Center A, with the Radius AG or AF, be drawn the Semicircle GFI; then, the right Line FI will be perpendicular to the Tangent aforesaid, the $\angle GFI$ being an Angle in a Semicircle; and HI will be equal to the Semiparameter AD; for $AF = AI$, and $KF = AH$, and consequently $HI = AK = AD$. From hence too, we may derive the Equation of the Curve found in the above *Solution*; for, by sim. Triangles, $GH : HF :: FH : HI$, and there-

fore $GH \times HI = HF^2$, that is, twice the Absciss multiplied into the Semiparameter, or the Absciss into the Parameter, is equal to the Square of the Semi-ordinate. — It may be observed also, that, if the indefinite right Line BM be drawn perpendicular to BA, then the Curve will be every-where equidistant from the said Line and the Focus A, i. e. $FM = FA$.

Quere the Nature of the Curve CFD, which is every-where equidistant from the Circumferences of the two unequal tangent Circles CE and CK, the latter being within the former? (Fig. 26.)

Solution.

Draw the Radius AE, and from the Center B the right Line BF, making $FK=FE$; then will the Point F be in the Curve in Quest. Now, $AF+FK=AE$, and therefore $AF+FK+KB=AE+BK=$ the Sum of the Radii of the two Circles. Hence, the Sum of the two right Lines drawn from the Foci of an Ellipsis to any Point in the Curve being always equal to its transverse Diameter, therefore the Curve in Question is an *Ellipsis*, whose Foci are the Centers, and transverse Diameter is the Sum of the Radii, of the two generating Circles. Q. E. I.

Note. If we put the Radius $AC=a$, Radius $BC=b$; then AB, the focal Distance $=a-b$; and (supposing the Point I the Center of the Ellipsis,) $BI=IA=\frac{a-b}{2}$, and $BF=AF=$

$$\frac{a+b}{2}. \text{ Now by 47. E. 1. } \overline{IF}^2 = \overline{AF}^2 - \overline{AI}^2 = \left| \frac{a+b}{2} \right|^2 - \left| \frac{a-b}{2} \right|^2 = ab, \therefore IF =$$

$\sqrt{ab}$: So that the Semiconjugate IF is a mean Proportional between the Radii BC and AC of the

the two generating Circles. — Again, put $HG=r$; then $BH=b+r$, and $AH=a-r$; and by 47 E. 1. $\overline{AH}^2 = \overline{HB}^2 + \overline{BA}^2$,
i. e. $\overline{a-r}^2 = \overline{b+r}^2 + \overline{a-b}^2$, *i. e.* $a^2 - 2ar + r^2 = b^2 + 2br + r^2 + a^2 - 2ab + b^2$; therefore, $2ab - 2b^2 = 2ar + 2br$, and $\frac{ab-b^2}{a+b} = r = HG$, to

which add $GB=b$, and we shall have $\frac{2ab}{a+b} = BH =$ the Semiparameter; a 3d Proportional to $\frac{a+b}{2}$ the Semitransverse, and $\overline{ab}^{\frac{1}{2}}$ the Semi-

conjugate. — It may also be observed, that, if $CN=CB$, and the Circle NM with the Radius AN be described; then the Curve will be every-where equidistant from the said Circle and the Focus B , *i. e.* $FM=FB$.

XI.

Quære the Nature of the Curve CF, which is every-where equidistant from the Circumferences of the two unequal tangent Circles CK and CE, the former being without the latter? (Fig. 27.)

Solution:

To any Point F in the Curve, draw the right Lines AF and BF from the Centers A and B . Now, because by Quest. $FE = FK$, therefore the Difference between the Lines AF and BF is always equal to the Difference between the Radii AE and BK , or CA and CB : Consequently, because the Difference of the two right Lines

I

drawn

drawn from the Foci of an Hyperbola to any Point in the Curve is always the same invariable Quantity, viz. its Transverse Diameter; therefore the Curve CF is an Hyperbola, whose Foci are the Centers B and A, and transverse Diameter is $CD = AC - BC$. Q. E. I.

Note. If the right Line GLT be drawn a Tangent to the two generating Circles; and from the Point C the perpendicular CH be erected, and through the Point H the indefinite right Line IO be drawn perpendicular to GT, and with the Radius IH the Semicircle AHB be described; then, the Subtangent of the Angle HIB, viz. CT, will be equal to the Semiparameter Bf, and CH (the Sine of the said Angle) equal to the Semiconjugate, and IC (the Cosine of the said Angle) will be the Semitransverse Diameter; and consequently the Line IO the Asymptote. The Point T is thus found; viz. Put $AC = a$, $BC = b$, and $TC = q$; then $TA : AG :: TB : BL$, i. e. $q + a : a :: q - b : b$, $\therefore qb + ab = qa - ab$, and $q = \frac{2ab}{a - b} = CT$.* Again, as HI is parallel to GA and LB, so it must bisect the Line AB, for H evidently bisects GL; and therefore, $IH = IB = \frac{a + b}{2}$, and $IC = \frac{a - b}{2}$, and consequently (by 47 E. 1.)

$CH = \sqrt{ab}$. Hence it appears, that, the Semiconjugate Diameter is a mean Proportional between

* See Quest. 12. Corol. 1.

between the Radii of the two generating Circles, and likewise between the Semitransverse and Semiparameter. ——— It may be observed also, that, if we make $CN = CB$, and with the Radius AN describe the Circle NM ; then the Curve, in any Point, will be equidistant from the said Circle and Focus B ; *i. e.* $FM = FB$.

SCHOLIUM.

FROM the three foregoing Questions, &c. it appears, that, the three conic Sections, *viz.* Parabola, Ellipsis, and Hyperbola, considering them as geometrical Curves, or without any Regard to the Solid out of which they may all be cut, have something uniform in their most simple and fundamental Properties, *viz.*

A *Parabola* is every-where equidistant from the Circumference of a Circle and a right Line passing through its Center; or from a Point and a right Line given in Position.

An *Ellipsis* is every-where equidistant from the Circumferences of two unequal tangent Circles which touch one the other *within*; or from the Circumference of a Circle and a Point *within* it.

An *Hyperbola* is every-where equidistant from the Circumferences of two unequal tangent Circles which touch one the other *without*; or from the Circumference of a Circle and a Point *without* it.

FROM the two last Solutions it likewise appears, that, if two unequal Circles intersect each other, as DM and EG , (*Fig. 28.*) whose Centers are A and B ; and a Curve, as QH , be

drawn equidistant from the said Circles, viz. $IG = IL$, $HM = HN$; then the said Curve will be an *Ellipsis*: And, if the Curve CK be drawn equidistant from the same two Circles, as $FN = FL$, $KP = KO$; then the said Curve will be an *Hyperbola*.

XII.

In the Hyperbola CF , (Fig. 27.) let any right Line OP be perpendicular to the Axis CP ; then will the Difference of the Squares of the said right Line and Semi-ordinate PF always be equal to the Rectangle of the Radii of the two generating Circles or Square of the Semiconjugate CH ; and consequently the Curve and Asymptote can never touch though infinitely produced. Quære the Demonstration?

Demonstration.

Put $AC = a$, $BC = b$, $EF = FK = v$, $CP = x$, $PF = y$, and $PO = z$. By sim. Δ s, $IC : CH :: IP : PO$; that is, $\frac{a-b}{2} : \overline{ab}^{\frac{1}{2}} :: \frac{a-b}{2} + x : z$; therefore $z = \overline{ab}^{\frac{1}{2}} + \frac{2 \cdot \overline{ab}^{\frac{1}{2}} x}{a-b}$, and $z^2 = ab + \frac{4abx}{a-b} + \frac{4abx^2}{a-b^2}$. By 47 E. 1. $\overline{AF}^2 - \overline{AP}^2 = \overline{PF}^2 = \overline{BF}^2 - \overline{BP}^2$; that is, $\overline{a+v}^2 - \overline{a+x}^2 = y^2 = \overline{b+v}^2 - \overline{x-b}^2$, or, $2av + v^2 - 2ax - x^2 = 2bv + v^2 - x^2 + 2bx$; therefore

therefore $v = \frac{a+b}{a-b}x$; which substituted for v

in $\overline{b+v}^2$, makes $y^2 = \overline{b + \frac{a+b}{a-b}x}^2 - \overline{x-b}^2$

$$= 2bx \frac{a+b}{a-b} + \frac{a+b}{a-b}^2 x^2 - x^2 + 2bx =$$

$$\frac{a+b}{a-b} + 1 \cdot 2bx + \frac{a+b}{a-b}^2 x^2 - 1 \cdot x^2 = \frac{4abx}{a-b} +$$

$$\frac{4abx^2}{a-b^2}. \text{ Hence, } z^2 - y^2 = ab, \text{ or (Quest. II.}$$

Note.) $\overline{PO}^2 - \overline{PF}^2 = \overline{CH}^2$. Q. E. D.

Corollaries.

1. When $x = b$, then $v = \frac{ab+b^2}{a-b}$; and

therefore $v+b = \frac{2ab}{a-b} =$ the Semiparameter

Bf. (See Quest. II. Note.)

2. Since $z^2 - y^2 = ab$; therefore $z+y : a :: b : z-y$.

XIII.*

Let the Quadrant ABC be completed into a Square ABKC; and let the Lines FG and fg be drawn parallel to AB and indefinitely near to each other; and from the Points E and e, where they intersect the quadrantal Arch, let the Perpendiculars

* Written in the Year 1756.

lars ED and ed be dropt on AB; and draw the Radius AE. Then, let the whole Figure revolve round the Axis AC, so as that the Quadrant shall describe an Hemisphere, and the Square a Cylinder circumscribing it. Upon which, I say, that, the infinitely narrow Rings described by the Line Ee, of which the whole spheric Surface is composed, are severally equal to the corresponding narrow Rings described by the Line Ff, which compose the whole cylindric Surface described by the Line BK; and consequently, the spheric and cylindric Surfaces are equal. Quære the Demonstration? (Fig. 29.)

Demonstration.

By the Figure, $En = Ff$, and the Triangles eEn and EAD are similar; consequently, as $eE : En (= Ff) :: EA : AD$, i. e. Ee is to Ff every-where as Radius to Cosine of BE , or as FG to EG ; but, FG and EG are the Radii of those circular Parallelograms, or Rings, described by the Lines Ff and Ee in the Revolution as above. So that, the Circumferences of Circles being as their Radii, the Lengths of the aforesaid corresponding Parallelograms will be every-where reciprocal to their Breadths, and consequently the said Parallelograms will be equal, &c. Q. E. D.

Corollaries.

1. As the curve Surface of a right Cylinder, whose Altitude is equal to the Diameter of its Base, is equal to the Circumference of that Base multiplied into its Diameter, which is 4 Times the

the

the Area of that Base, so the Surface of a Sphere must be 4 Times the Area of a great Circle of the said Sphere.

2. The convex Surface of any Portion of a Sphere, as CE (revolved), is equal to the cylindric Surface FK (revolved,) i. e. equal to FK multiplied into the Circumference of a great Circle of the said Sphere, or, which is the same, equal to the Circumference of a great Circle multiplied into CG the versed Sine of the Segment whose spheric Surface is required.

3. The whole spheric Surface may be projected on the Concave of a right cylindric Tube that circumscribes it, by Perpendiculars from the Axis AC, as GF; so that the Surface Ee &c. will be transferred to Ff &c. And the said Tube being opened and extended on a Plain, may exhibit a Chart of the whole teraqueous Globe; having this peculiar Property, viz. that, the Surface of the said Chart will be exactly equal to the globular Surface which it represents.

N. B. According to this Projection, the Meridian, from the Equator to the Pole, would be thrown into a Line of Sines.

XIV.

Required to draw a Tangent to the common Cycloid geometrically. (Fig. 30.)

Solution.

On the Diameter AC let there be erected a Perpendicular CD equal to the semicircular Circumference

circumference AEC; and let the said Semicircle move on equably upon the said Line CD, so as that the Diameter AC continue always in a parallel Situation, that is, always perpendicular to the Line CD; and at the same Time, let a Point move round, equably, in the said semicircular Arch, from A to C: then will the said Point describe the Semi-cycloid AFD. Now, the generating Point is carried with a compound Motion, *viz.* a circular Motion in the generating Circle AEC, and a progressive Motion in the constant Direction Bb or CD; and these two Motions are every-where of an equal Velocity; for, whilst the said Point is moving through the Semicircumference AEC, the Semicircle itself is moving through the same Distance, *viz.* CD. From hence, then, may be easily found the Direction of the Cycloid in any Point, as F or *f*, or, which is the same, a Tangent to the Cycloid in that Point; for the generating Point does there proceed in the Circle's Circumference according to the Direction TF or *tf* (the Tangent to the Circle in that Point,) and the Circle itself carries the Point in the Direction EF or *ef* with an equal Velocity: Consequently, the Direction of the Cycloid (or its Tangent) in that Point, must be exactly in the Mid-way between the two former, that is, in the Chord *aF* or *af* which bisects the Angle TFE or *tfe*. So that the Tangent to any Point in the Cycloid, as F, is always parallel to the corresponding Chord, as AE, of the generating Semicircle AEC.

Q. E. F.

Note. The Direction of the Tangent to the Cycloid at any Point, is the same as the Direction of the Chord of the generating Circle, which passes through that Point.

Note. That the Chord aF bisects the Angle EFT , is evident; for $\angle bFT = \angle cFa =$ a right Angle, and therefore $\angle aFT = \angle bFc = \angle bcF = \angle EFa$.

Corollary.

From the above it follows, that, the Arch $AE = EF$, or Arch $Ae = ef$; for, as far as the generating Point moves round in its circular Orbit, so far that Orbit itself, and every Point of it, moves onward in its straight Course Bb .

XV.

Required to draw a Tangent to the interior or inflected Cycloid geometrically. (Fig. 31.)

Solution.

Let CD be equal and parallel to GH , and equal to the semicircular Circumference AC ; and let the common Semicycloid AD be generated as before in Prob. 14. and at the same Time, let a Point move uniformly along the concentric Semicircle EeG from E to G ; then will the said Point describe or generate the inflected Semicycloid EFH . Now to draw a Tangent to any Point F in the inflected Cycloid, we need only to consider the Proportion of the Radii of the con-generating Circles, *viz.* BA and BE ; for, the progressive Velocity of the Point generating the inflected Cycloid is evidently equal to that of the Point generating the common Cycloid; and the circular Velocity of

K

the

the former is to that of the latter as BE to BA : Consequently, the Direction of the inflected Cycloid in any Point, as F, (or its Tangent in that Point,) coincides with the Diagonal of a Parallelogram, whose Side FI or Ta represents the progressive, and FT or Ia the circular Velocity and Direction of the generating Point F; that is, let FI be parallel to GH and equal to BA, and Ia be equal to BE and parallel to the Tangent *mn*; then aF will be the Tangent required.

Corollary.

From the above it follows, that, Semicircle E_nG : Base GH :: Arch E_n : nF ; which, indeed, is the well-known Property of every Cycloid whose Base is a right Line.

XVI.

Required to find the Point of Inflection in an inflected Cycloid, without a fluxional Operation, (Fig. 32.)

Solution.

Let the right Line CD be tangent and equal to the semicircular Circumference CfA, to which let there be another concentric Semicircle GtE; and let the said greater Semicircle AfC turn on the Base CD, carrying the lesser Semicircle E/G along with it: Then, the Point A will describe the common Cycloid AD, and the Point E will describe the inflected Cycloid EeH. Now,

Now, a right Line drawn *from* the Point where the great generating Circle touches the Base to a generating Point carried along with that Circle, is always perpendicular to the Curve described by the said generating Point; that is, Fe is perpendicular to the inflected Cycloid at any Point e : For, it is very evident, from the Contemplation of the Fig. that the Motion of the Line Fe , on the Center F , must describe some small Portion of a circular Circumference at e , which must be perpendicular to its Radius Fe ; or, there being only one Point F of the generating Circle cFa that touches the Base at one Time, every Line issuing from the said Point F (in the finite moving Plane cFa) must have an angular Turn, or Motion, before the said Point F moves off from the Base, and consequently will describe with its other Extremity, *viz.* e , a small circular Arch: Therefore, &c. From hence it follows, that, the Curve described by the Point E must have a Point of Inflection; for, when the Point A or a arrives at D , Ee will coincide with DH , and HG will be a Tangent to the Curve at the Point H ; and consequently, the Tangent to the Point E being parallel to GH , the Curve must be concave at E and convex at H . — Again, it is evident, that, at the Point of Inflection, the Tangent has the greatest Elevation, or makes the greatest Angle with the Base CD ; and that from that Point, towards either Extreme, the said Tangent is continually depressed, till at the Point E or H it comes to a parallel Situation with the Base CD . Draw Ef perpendicular to the Diameter AC : then, (it follows from what has been said,)

when the Point f arrives at the Base CD , the Tangent to the Curve will coincide with the Diameter EG . Now, it is plain, that, in the Generation of the Curve EH , its Direction will be *more* elevated than the Diameter of its generating Circle, before the said Diameter becomes a Tangent to it, and afterwards *less*: For, let Fc be perpendicular to the Diameter ac ; and the Points m and n infinitely near to F ; then, the Angle mec will be *less* and the Angle nec *more* than a right Angle; but, when either of the Points m or n is in the Base CD , the right Line me or ne will be perpendicular to the Curve: Therefore, &c. Hence then it appears, that, when the Point f arrives at the Base CD , the Diameter becomes a Tangent to the Curve, and passes through the Point of Inflection: And therefore, to find the said Point of Inflection; draw the Tangent Ct from the lower Extremity of the larger Semicircle to the lesser, and through t draw the right Line te parallel to the Base; and it will cut the inflected Cycloid in the Point of Inflection e required.

Or thus, by the Editor. (Fig. 33.)

* Put the generating Semicircle $AGF = a$,
Base $FD = b$, Radius $OG = c$, and $OC = x$.
Let

* This Curve may be thus generated, *viz.*

1°. Let a Semicircle af (Fig. 33.) be rolled along upon a right Line fd equal to its Circumference and perpendicular to its Diameter af ; then will the Curve ABD , described by any Point A taken within the said Semicircle and in the Radius Oa , be an inflected Semicycloid. — For, draw the concentric Semicircle AGF ; in any Position of which, as BRK , to the generating Point B draw the Ordinate CB parallel to the Base FD , which said Base must evidently be equal and parallel to fd .

Let gb be supposed indefinitely near and parallel to GB . Now, it is evident, that, at the Point of Inflection, the Angle made by the Tangent and Ordinate must be a Maximum; that is, at the Point of Inflection B , the $\angle Bbv = a$ Maximum. But, by the Nature of the Curve, $a : b (:: AG : GB :: AG + Gg : gv + vb$, and therefore, GB being $= gv$, and $Gg = Bv$,) $:: Bv : vb$; and by Trigonometry, $bv : vB :: s\angle vBb : s\angle Bbv$; therefore the said $\angle Bbv$ is the greatest when vB is perpendicular to Bb . Consequently, at the Point of Inflection B , the Tangent

fd : through the Center o draw Mr parallel to the Diameter af ; and with the Radius oB describe the Arch BM . Then will the Arches AG , MB , and RK , be equal, and $oB = CG$; and therefore $GB = Cr = fr =$ (by the Generation) Arch rk . But Semicircle AF : Semicircle $af ::$ Arch RK : Arch rk . Hence therefore, Semicircle AF : fd or Base $FD ::$ Arch AG : GB ; which is the Property of the Curve in Question. Therefore, &c.

20. *Or thus.* In Fig. 34. Let the circular Arch FE roll uniformly along the right Line FD , which is equal to it and perpendicular to the Diameter AF of the Semicircle AGF ; and at the same Time let a Point move with an uniform Motion from A along the Arch AE , and such Velocity as to arrive at E the very Instant the said Point E arrives at D : then will the Curve described by the Point moving from A to E be an inflected Semicycloid ABD . — For in any Position aRf of the Semicircle AGF , to the generating Point B draw the Ordinate CB parallel to the Base FD ; draw also MR equal and parallel to the Diameter AF ; and with the Radius oa describe the Arch aM . Then, $FE : AE :: fR : aB$, and therefore $FE - AE : FE :: fR - aB : fR$. But, $fR - aB = MB = AG$, and $fR = FR = Cr = GB$. Consequently, Semicircle AGF : FE or Base $FD ::$ Arch AG : GB . Therefore, &c.

30. *Or thus.* In Fig. 34. let the Semicircle AGF be less than the Base or right Line FD perpendicular to its Diameter AF ; along which said right Line let it be carried with a parallel and uniform Motion; and at the same Time, suppose a Point to move uniformly from A along the said Semicircle, and with such Velocity as to arrive at F the very Instant the said Point F arrives at D : then will the Point moving along the Semicircle AGF describe an inflected Semicycloid ABD . — For, in any Situation MBR of the said Semicircle, let CB be parallel to the Base FD ; then, B being the generating Point, Semicircle AGF : Base $FD ::$ Arch MB : FR . But $MB = AG$, and $FR = Cr = GB$. Consequently, Semicircle AGF : Base $FD ::$ Arch AG : GB . Therefore, &c.

Tangent is coincident with the Radius oB ; and the Triangles Bvb and roB or COG are similar; and therefore, $Bv : vb :: CO : OG$; *i. e.*

$$x : b :: a : c, \therefore x = \frac{ac}{b} = OC.$$

Corollaries.

1. At any Point B in the Curve, if to the corresponding Point G in the Circle AGF , we draw the right Line TG perpendicular to the Radius OG , making $TG : GB :: a : b$; that is, if TG be made equal to the Arch AG ; or if BE be made $=$ the Radius Oa , and EI be drawn parallel to the Tangent TG and equal to the Radius OA ; then will the right Line drawn from T or I to B , be a Tangent to the Cycloid at the Point B . — For then, the Triangles TGB or IEB and Bvb will be similar; and therefore, &c.

2. If the Curve be supposed to be generated as in Note 1^o. and the right Line AQ be drawn perpendicular to the Radius Oa ; then, when the Point Q arrives at the right Line fd parallel to the Base FD , the Point A will be in the Point of Inflection B . — For, since at the

said Point of Inflection, $x = \frac{ac}{b}$; by Analogy,

$b : a :: c : x$, *i. e.* Semicircle prk : Semicircle $BRK :: oB : ot$; therefore, the Circumferences of Circles being as their Radii, $ra : oB :: Bo : ot$, *i. e.* ot or OC is a third Proportional to the Radii Of and OF , and the Triangles roB and Bot are similar. Consequently,

quently, rB is perpendicular to po ; and therefore, &c. — Hence the following Construction, *viz.* Make $DR = \text{Arch } fe$; draw rM through the Point R equal and parallel to fA ; make $oR = OF$; and lastly, draw the Semicircles MR and or : Then will the intersecting Point B of the said Semicircles be the Point of Inflection required.

XVII.

Required to find the Point of Retrogression in a curtate Cycloid, without a fluxional Operation.
(Fig. 35.)

Solution, by the Editor.

* Put the generating Semicircle $AGF = a$, Base $FD = b$, Radius $OG = c$, and $OC = x$.
Let

* This Curve may be thus generated, *viz.*

19. Let a Semicircle af (Fig. 35.) be rolled along upon a right Line fd equal to its Circumference and perpendicular to its Diameter af : then will the Curve ABD , described by any Point A taken without the said Semicircle, and in the Radius Oa produced, be a curtate or contracted Semicycloid. — For, draw the concentric Semicircle AGF ; in any Position of which, as BRK , to the generating Point B draw the Ordinate CB parallel to the Base FD , which said Base must evidently be equal and parallel to fd : through the Center o draw MR parallel to the Diameter AF ; and with the Radius oB describe the Arch BM . Then will the Arches AG , MB , and RK , be equal, and $rB = CG$; and therefore $GB = Cr = fr =$ (by the Generation) Arch $r\lambda$. But, Semicircle AF : Semicircle af :: Arch RK : Arch $r\lambda$. Hence therefore, Semicircle AF : fd or Base FD :: Arch AG : GB ; which is the Property of the Curve in Question. Therefore, &c.

20. Or thus. In Fig. 36. Let the circular Arch FE roll uniformly along the right Line FD , which is equal to it and perpendicular to the Diameter AF of the Semicircle AEF ; and at the same Time, let a Point move with an uniform Motion from A along the Arch AE , and such Velocity as to arrive at E the very Instant the said Point E arrives at D ; then will the Curve described by the Point moving from A to E be a curtate or contracted Semicycloid ABD . — For, in any Position aRf of

Let gb be supposed indefinitely near and parallel to GB . Now, it is evident, that, at the Point of Retrogression, the Tangent must be perpendicular to the Ordinate; that is, at the Point of Retrogression B , the Increment of the Curve, Bb , is perpendicular to bv ; and therefore, because the $\angle tBb = \angle oBv$, the right-angled Triangles Bbv and Bto or GCO are similar, and consequently $Bv : vb :: GO : OC$. But, by the Nature of the Curve, $a : b :: Gg$ or $Bv : vb$. Hence therefore, $a : b :: GO : OC$; *i. e.* $a : b :: c : x$; $x = \frac{bc}{a} = OC$.

Corollaries.

1. At any Point B in the Curve, if to the corresponding Point G in the Circle AGF , we draw the right Line TG perpendicular to the Radius OG , making $TG : GB :: a : b$; that is,

of the Semicircle AEF , to the generating Point B draw the Ordinate CB parallel to the Base FD ; draw also MR equal and parallel to the Diameter AF ; and with the Radius a describe the Arch AM . Then $FE : AE :: fR : aB$, and therefore $FE + EA : FE :: fR + aB : fR$. But, $fR + aB = MB = AG$, and $fR = FR = Cr = GB$. Consequently, Semicircle $AEF : FE$ or Base $FD :: Arch AG : GB$. Therefore, &c.

2^o. Or thus. In Fig. 36. Let the Semicircle AGF be greater than the Base or right Line FD perpendicular to its Diameter AF ; along which said right-Line let it be carried with a parallel and uniform Motion; and at the same Time, suppose a Point to move uniformly from A along the said Semicircle, and with such Velocity as to arrive at F the very Instant the said Point F arrives at D : then will the Point moving along the Semicircle AGF describe a curtate or contracted Semicycloid ABD . — For, in any Situation MBR of the said Semicircle, let CB be parallel to the Base FD ; then, B being the generating Point, Semicircle $AGF : Base FD :: Arch MB : FR$. But, $MB = AG$, and $FR = Cr = GB$. Consequently, Semicircle $AGF : Base FD :: Arch AG : GB$. Therefore, &c.

is, if TG be made equal to the Arch AG; or BE be made = the Radius Oa, and EI be drawn parallel to the Tangent TG and equal to the Radius OA; then will the right Line drawn from T or I to B, be a Tangent to the Cycloid at the Point B. — For then the Triangles TGB or IEB and Bvb will be similar; and therefore, &c.

2. If the Curve be supposed to be generated as in Note 1°. and the right Line aQ be drawn perpendicular to the Radius OA; then, when the Point Q arrives at the Base FD, the Point A will be in the Point of Retrogression, B. — For, since at the said Point of Retrogression, $x = \frac{bc}{a}$; by Analogy, $a : b :: c : x$; i. e. Semicircle BRK : Semicircle prk :: oR : ot; therefore, the Circumferences of Circles being as their Radii, the Point t coincides with r, i. e. the Ordinate CB coincides with the right Line fd; and the Triangles Rop and Bot are equal and similar. Consequently Rp is perpendicular to po; and therefore, &c. — Hence the following Construction, viz. Make DR = Arch ae; draw Rt equal and parallel to Ff; and tB equal and parallel to aQ: then will B be the Point of Retrogression required.

XVIII.

Let the Semicircle kd roll along upon the circular Arch df, which is equal to it, and carry with it the inner concentric Semicircle KD: then will the Point D describe the Semicycloid DA. Quære wheiber the said Cycloid be any-where convex towards the Center S; that is, whether the Point

L

T,

T, where the Tangent TG has the greatest Obliquity with the Line TS, be always a Point of Inflection; or, in other Words, whether the Part DT of the said Curve be convex towards the Base? (Fig. 37.)

Solution.

Let the generating Circle turn on the infinitely small Arch dr , and draw the Lines Dr , or , and Sr . Now, as the Tangent of the Cycloid is always perpendicular to rD , (r being the Point where the generating Circle touches the Base, and D the generating Point in the said Circle;) so, at the first setting out, it must be perpendicular to dD . The Question then is, whether the Tangent from that first setting out, begins to turn to the *right*, or to the *left* Hand, (supposing it to turn like the Hand of a Clock on its Tangent Point as a Center;) if the *former*, then the Curve is evidently *convex* (at the first setting out,) towards the Base; if the *latter*, it is *concave*. Now, when the generating Circle has turned from d to r , the Tangent will have turned to the *left* by the Quantity of the Angles rod and rSd ; and to the *right* by the Quantity of the Angle rDd . So that the Question will terminate in this, *viz.* which is greatest, the Angle rDd or the Angles $rod + rSd$? — if the former, the Curve is convex, if the latter, concave. — Now, if ro or $do = a$, and rS or $dS = b$, and $oD = c$; then $\angle rSd : \angle rod :: a : b$, $\therefore \angle rSd = \frac{a}{b} \angle rod$. Again, $\angle rDd : \angle rod :: ro : rD$ or $dD :: a : c$;

: $a-c$, $\therefore \angle rDd = \frac{a}{a-c} \angle rod$. But, $\angle rSd$

+ $\angle rod = \frac{a}{b} + 1 \angle rod = \frac{a+b}{b} \angle rod$. Con-

sequently, in Order to a Convexity at D, or a Point of Inflection at T, $\frac{a}{a-c}$ must be more

than $\frac{a+b}{b}$; that is, (multiplying by the Deno-

minators $a-c$ and b ,) ab must be more than $a^2 - ac + ab - bc$; and therefore, by Transpo-

sition, $ac + bc$ must be more than a^2 ; and con-

sequently c must be more than $\frac{a^2}{a+b}$; or, c must be greater than a third Proportional to $a+b$ and a , in Order to make T a Point of Inflection, or for the Cycloid to be any-where convex towards the Base. Q. E. I.

XIX.

When an interior Cycloid upon a convex Base, as ABD, is of such a Nature as to have a Point of Inflection, as B; it is required to find that Point; or, to determine how far the generating Circle shall have rolled on when it passes that Point?

— Or, to speak more distinctly; where does the generating Circle stand upon its Base, when the generating Point of the Circle passes the Point of Inflection? (Fig. 38.)

L 2

Solution.

Solution.

The Tangent at the Point of Inflection B has evidently the greatest Degree of Obliquity with the Line BS: And, as the said Tangent is always perpendicular to the Line B γ drawn from the generating Point in the Diameter of the moving Circle to the Point where the said Circle touches its immoveable Base; so the Point of Inflection required must be where the Angle γ BS is a *Maximum*, (BS being a right Line drawn from the generating Point to the Center of the Base;) for the greater the Angle γ BS is, the more oblique will the Perpendicular of γ B be to BS. The whole Business then is reduced to this, *viz.* Given the Radii OF, Of, and fS; required the Point P in the Periphery FPA, where the Angle fPS is a *Maximum*?—And this ought to be done in a distinct Problem and Figure for the Purpose. (*See Prob. 20. Chap. 1.*)

N. B. When the Arch FP or RB is found, the rest is very obvious; for the Arch RB, or (which is the same) γp , subtracted from 180° , leaves the Arch γk , which will discover γf by a Comparison of the Radii γo , and γS ; for, as $\gamma S : \gamma o ::$ Degrees of $\gamma k : \text{Degrees of } \gamma f$.
Q. E. Q.

SCHOLIUM.

If the Orbits of the Moon round the Earth and the Earth round the Sun be supposed to be Circles, and their Planes to coincide; then,
from

from Prob. 18. it will follow, that the Curve described by the Center of the Moon in one Luration, is no-where *convex* towards the Sun; and, by the last Prob. may be determined *that* Point in the said Curve where it has the *greatest* Obliquity towards it.

XX.*

If a rectangular Piece of Paper be turned into the Form of a right cylindrical Tube, and a Sphere be inscribed therein, so as that the Axes of the Sphere and Cylinder do coincide, or, that the Equator be the Line of Contact between the said Tube and Sphere, and all the Points of the spheric Surface be projected or transferred to the concave Surface of the Tube, by right Lines proceeding from the Center of the Sphere, and terminating in the said concave Surface of the Tube: And then, if the Paper be opened and stretched upon a Plane, it will present a Chart, in which the Meridians, Parallels of Latitude, and Rhumbs, are all truly and geometrically projected in right Lines; as in Fig. 39. Quære the Demonstration?

Demonstration.

With Regard to the Meridians, it is evident, that, they are all thrown into right Lines in the Tube, being all parallel to its Axis: And, as the Parallels of Latitude are all projected in Circles perpendicular to the said Meridians; so, upon opening the Tube, &c. as aforesaid, they

* Written in the Year 1746.

they must necessarily become right Lines also. The only Thing therefore that requires a Demonstration, is, that the Rhumbs, or Loxodromics, become right Lines when the Paper Tube is extended as above. In Order to this, let the Eye be supposed to be placed in the Center of the Sphere when inscribed in the Tube; then every Rhumb will appear to run round the concave Tube in the Manner of a Bottle-screw *in infinitum*. And the only Thing to be proved, is, that it keeps a parallel Direction to itself every-where; or, that it makes the same Angle with all the Meridians; or, that the projected Rhumb makes the same Angle with the projected Meridian, as the true Rhumb makes with the true Meridian upon the Surface of the Sphere. These two Angles do apparently coincide, with Regard to the Eye placed as aforesaid; that is, they are *apparently* equal, to the Eye in that Situation; and that they are also *really* equal, is evident from this Lemma, *viz.* That the *real* and *apparent* Bigness of any Angle are the same when the Eye is placed perpendicularly over either of its Sides, or, when a Perpendicular dropt from the Eye to the Plane of the Angle falls upon either of its Sides. Now, this is the very Case with Regard to both the Angles in Question; for, the Perpendicular from the Eye falls on the angular Point of the Angle on the Sphere; and a Perpendicular from the Eye falls on the Meridian, which is one Side of the Angle on the Tube: Consequently, the real and apparent Bigness of each of those Angles is the same; and therefore, as they appear equal, they are really so. Q. E. D.

Scholium.

Scholium.

It does not appear that *Mercator*, or *Wright*, ever thought of this Projection; for, the Meridian Line here is manifestly a Line of *Tangents*; whereas, in their Projection, it is a *Collection* of *Secants*. It may be added, that, *Mercator's* or *Wright's* Chart, is very faulty in the Bearing of Places; but in this, it is as true and correct as upon the Globe itself. I shall therefore presume to say, that this naval Planisphere, or Sea Chart, is the most useful for the Purposes of Navigation ever yet invented; it being better than *Mercator's* in one important Respect, and equal to it in all others.

THERE are three Projections of the Sphere, the orthographic, the stereographic, and the nautical; the two first of these are well known to Mathematicians: The last was invented for the Purposes of Navigation, though hitherto a very imperfect and defective Invention. The Errors of the plain Chart are corrected, in a great Measure, by *Mercator's* or *Wright's* Chart; though this latter is not a *true* Projection of the Sphere in any Shape; nor indeed is it pretended to be such by Mr. *Wright* one of its Inventors, who represents it rather to be an Extension of the spherical Surface upon the inner Side of the concave Cylinder in which it is inclosed. Suppose (*c. g.*) the Globe to be so inscribed in a cylindric Tube as to touch it every-where in the Equator, and consequently the Axes of the Globe and Cylinder to coincide; then, suppose the Tube

to be of hard and unyielding Substance, as of Marble or the like, and the Globe to be of a soft Substance, as a Bladder, and to enlarge itself as that does when blown, until the globular Surface becomes a cylindrical one by applying itself to the internal or concave Surface of the Cylinder, both Ways towards each Pole; Mr. *Wright* supposes, all the Parts of the spherical Surface to increase uniformly in this Extension; or, so as that the Degrees of Longitude and Latitude every-where shall still continue to bear the same just Proportion to each other, *i. e.* as Radius to Secant of Latitude

— Whereas, the true Projection, (and which I apprehend will much better answer the Purposes of Navigation than either the plain Chart or Mr. *Wright's*,) is this, *viz.* Let the Sphere be inscribed in a cylindric Tube, as above; and let all the Parts of the spheric Surface be transferred to the concave cylindric Surface, by right Lines drawn from the Center of the Sphere: The Consequence of which is, that, when the Cylinder is opened and spread upon a Plain, the Meridians, Parallels, and Loxodromics, will be all projected in right Lines, as in *Mercator's* or *Wright's* Chart, but in different Proportions. And, I take upon me to assert, that this is the first Chart, or Representation of the terraqueous Globe, ever yet invented, in which the *Meridians*, *Parallels*, and *Rhumbs*, are justly and truly projected in right Lines; for the latter cannot be so projected in *Mercator*.

XXI.

If the Quadrant of a Meridian AC, viz. from the Pole to the Equator, be projected into a Line of Tangents, as CD &c. for the Purpose of a naval Chart, (as in the foregoing Quest.) in which all the Parallels of Latitude are severally equal to a great Circle and to one another; I say, that, the Increments of Latitude, in that Projection, will be to the corresponding Increments of Longitude, in the duplicate Proportion of the said Increments in Mercator's or Wright's Chart, i. e. the duplicate Proportion of the Secant to the Radius. *Quare the Demonstration?* (Fig. 40.)

Demonstration.

Let the prick'd Line BF be drawn infinitely near to the Secant BD; and let the Arch HF be concentric to GI: then GI will be equal to the Increment of Longitude (being the Increment of a great Circle) corresponding to FD the Increment of Latitude. But GI is to HF as Radius to Secant, and HF is to FD evidently in the same Ratio, (both from the Similarity of Triangles;) Consequently, GI is to FD as the Square of Radius to the Square of the Secant. Q. E. D.

XXII.

To find the common Center of Gravity of the three Squares erected upon the three Sides of a right angled

M

angled

angled Triangle ABC. — Make CD equal and parallel to BA, and DE perpendicular to CA; and from E let fall EF perpendicular upon GB; draw the right Line FH: Then will the bisecting Point P of this Line be the common Center of Gravity required. Quere the Demonstration? (Fig. 41.)

Demonstration, by the Editor.

It is evident, that, the Centers of Gravity of the three Squares respectively, are the Points H, I, and K; and, that, if IF be made to FK as the Square LB to the Square BG, then the Point F will be the common Center of Gravity of the said two Squares: Therefore, since these two Squares are exactly equal to the other Square, the Point P which is equidistant from the Center of Gravity of the greatest Square and the common Center of Gravity of the other two, must be the common Center of Gravity of all the three Squares.

By sim. Δ s, $EC : CD :: CD : CA$, and $EA : AD :: AD : CA$; $\therefore CA = \frac{CD^2}{EC} = \frac{AD^2}{EA}$, and therefore $EC : EA :: CD^2 : AD^2$; but $CE : EA :: IF : FK$; $\therefore IF : FK :: CD^2 : AD^2$; that is, IF is to FK as the Square LB is to the Square BG. Therefore, &c. Q. E. D.

Corollary.

Corollary.

In any right-angled plain Triangle, if a Perpendicular be let fall from the right Angle upon the Hypothenuſe, the Segments of the Hypothenuſe will be as the Squares of the two Legs.

XXIII.

Required the Content of a Solid contained under 8 regular and equal Hexagons and 6 equal Squares; the Sides of the Hexagons and Squares being equal. (Fig. 42 and 43.)

Solution.

From an Octahedron, cut off its 6 ſolid Angles at $\frac{1}{3}$ of their Depth; then will the Remainder of it be the Exoctahedron or Body in Queſtion. Join 2 of theſe ſolid Angles cut off, together; and they will evidently form an Octahedron, whoſe Side is $\frac{1}{3}$ of the Side of the Octahedron from which it is cut; and therefore, ſimilar Solids being as the Cubes of their homologous Sides, the ſaid little Octahedron will be to the greater as 1 to 27. Conſequently, the 6 ſolid Angles cut off will be $\frac{3}{27}$ ths or $\frac{1}{9}$ th of the ſaid great Octahedron; and therefore, $\frac{6}{9}$ ths of it are the Content of the Body in Queſtion.
Q. E. I.

XXIV.

Required the Content of a Solid contained under 20 regular and equal Hexagons and 12 regular and equal Pentagons; the Sides of the Hexagons and Pentagons being equal. (Fig. 44 and 45.)

Solution.

This Solid is formed, by cutting off the 12 solid Angles of an Eicosihedron at $\frac{1}{3}$ of their Depth; for then, its 20 triangular Faces will be turned into so many Hexagons, and the Bases of the said solid Angles cut off will be regular Pentagons. Now, because like Solids are in the triplicate Proportion of their corresponding Dimensions; therefore, each of the pentagonal Pyramids, or solid Angles cut off, is to the pentagonal Pyramid, or solid Angle from which it is cut, as 1 to 27; and consequently, $\frac{1}{27}$ ths or $\frac{4}{3}$ ths of the said larger Pyramid are equal to all the solid Angles cut off; which therefore being subtracted out of the Content of the Eicosihedron aforesaid, leave the Content of the Eiccosidodecahedron or Body in Question. Q. E. I.

XXV.

Required the Proportion of a Tetrabedron to an Octahedron, when their Sides are equal. (Fig. 46.)

Solution.

Solution.

From the Tetrahedron ABCD, cut off all the solid Angles at half the Depth of the whole Solid; the Result will be the Octahedron EFGHIK. Now, similar Solids being as the Cubes of their like Dimensions, the Tetrahedron BEFG, whose Side is half the Side of the Tetrahedron ABCD, will be to it as the Cubes of those Sides, *i. e.* as 1 to 8: And therefore, as 4 of those lesser Tetrahedrons have been cut off from the greater; so, 4 more must remain in the Octahedron aforesaid; and consequently, the said Octahedron is to a Tetrahedron of an equal Side with it, as 4 to 1. *Q. E. D.*

F I N I S.



ERRATA

P. 12. l. 4. r. (multiplying by $4ax - 4x^2$ ¹
and dividing by x)

P. 35. l. penult. f. AD, *i. e. ae*, r. AE is
equal to the Fluxion (or Swiftneſs of Decrease)
of the Line AD, *i. e. ae*

P. 67. l. 22. f. Ea, r. Fe

March 25, 1762.

In the Press, and speedily will be published,
For the Benefit of the Author's WIDOW,
In one Volume, Octavo, Price, in Boards, 5s.
DISCOURSES on various interesting and
important Subjects.

By the late Rev. Mr. WILLIAM WEST.

Lately published,

In one Volume, Octavo, Price, bound, 4s.
The Nature, Design, Tendency, and Importance of Prayer: Illustrated in Seven practical Dissertations on the LORD'S PRAYER.

By the same AUTHOR.

March 22, 1762.

in the Field, and speedily will be published,
for the Benefit of the Author's Widow,
In one Volume, Octavo, Price, in Boards, 4s.
Just Published,
DISCOURSES, on various interesting and

In one Volume, Octavo, Price, bound, 4s.
By the late Rev. Mr. WILLIAM WEST
**An Introduction to the DOCTRINE of
FLUXIONS.**

**The Second Edition, with Additions and Al-
terations.**

In one Volume, Octavo, Price, bound, 4s.
By JOHN ROWE.
The Nature, Design, Reasons, and Impor-
tance of Prayer: illustrated in seven practi-
cal Dissertations on the Lord's Prayer.
By the same Author.

Plate I.

Fig. 1.

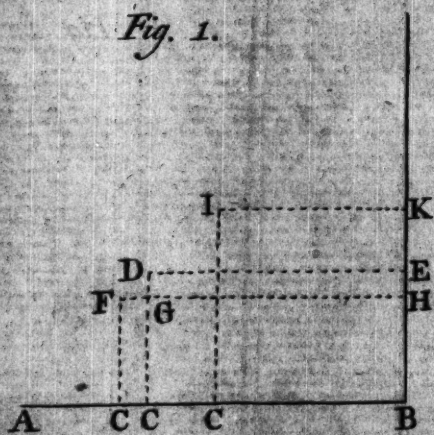


Fig. 2.

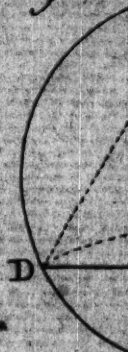


Fig. 3.

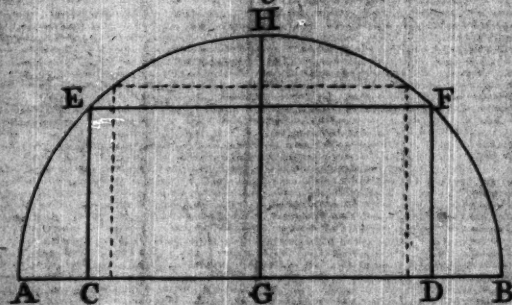


Fig. 6.

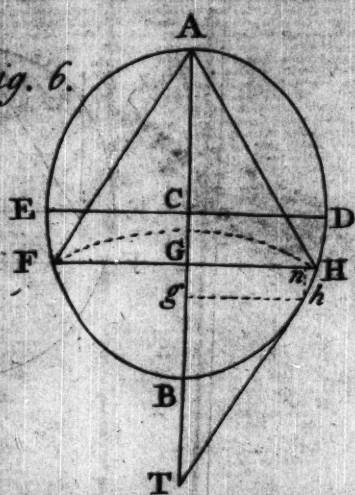


Fig. 7.



Fig. 1. to 8.

Fig. 2.

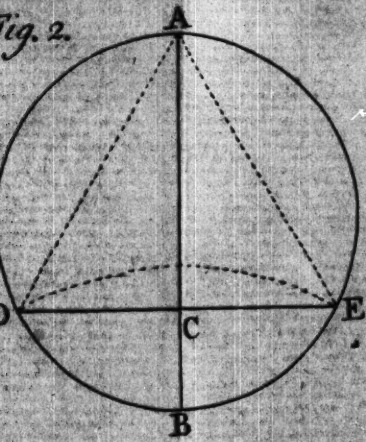


Fig. 4.

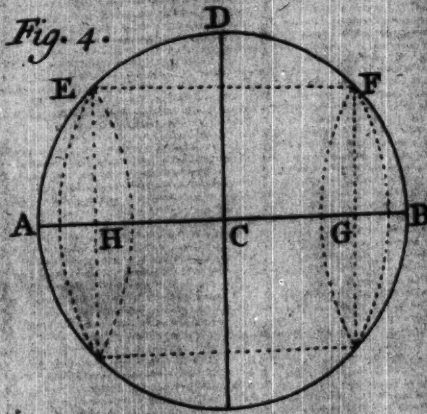


Fig. 5.

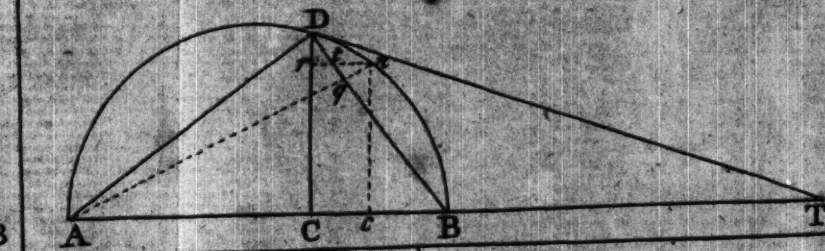


Fig. 7.

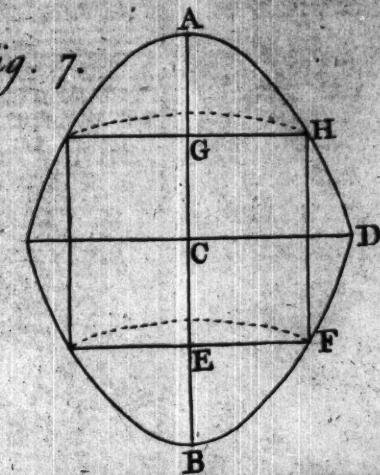
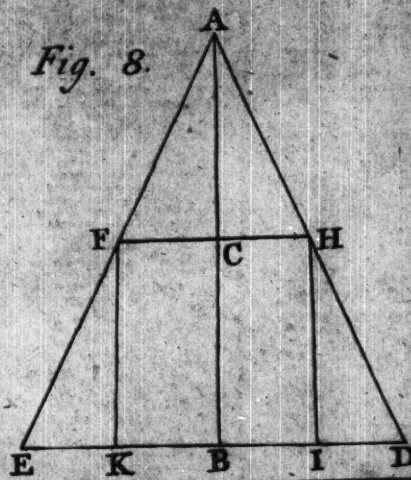


Fig. 8.



L. Mynde Sr.

Plate II.

Fig. 9.

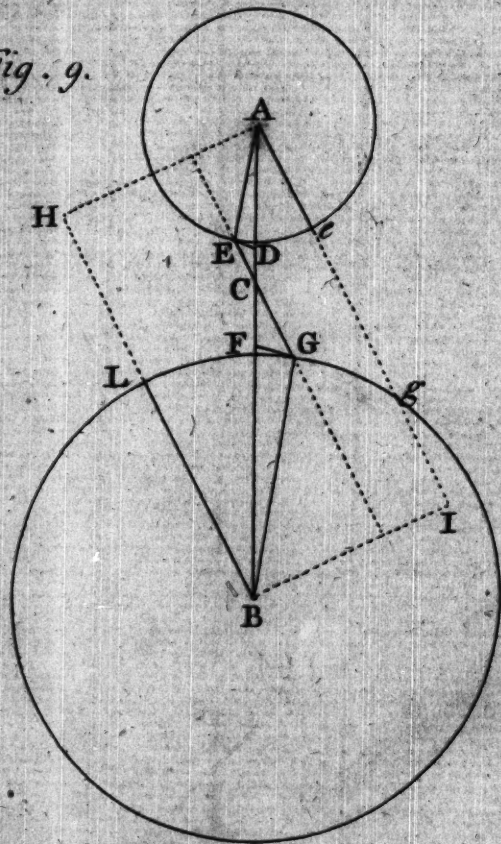
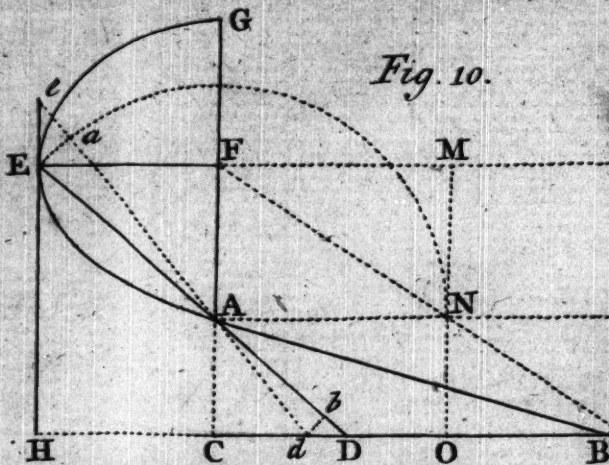


Fig. 10.



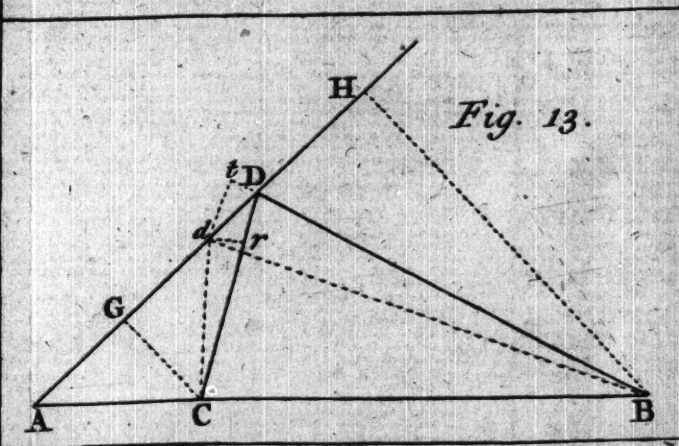
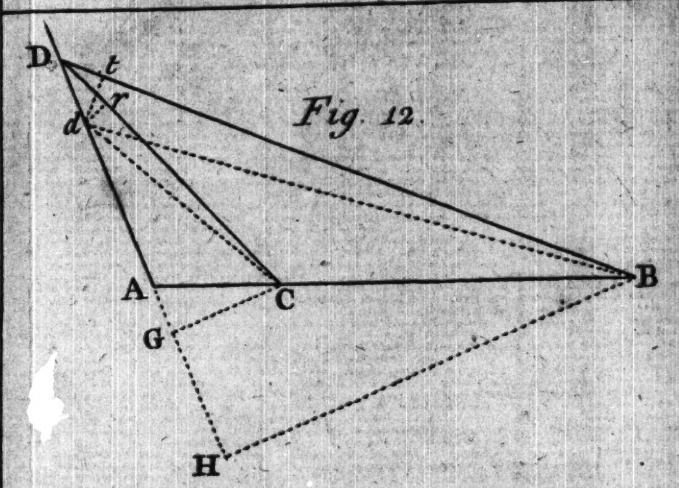
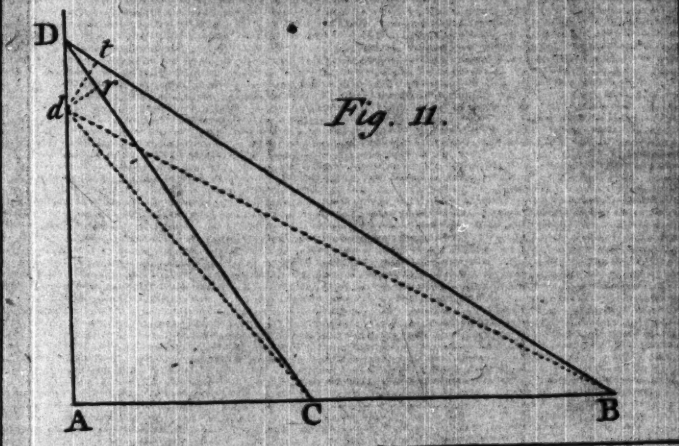


Plate III.

Fig. 14.

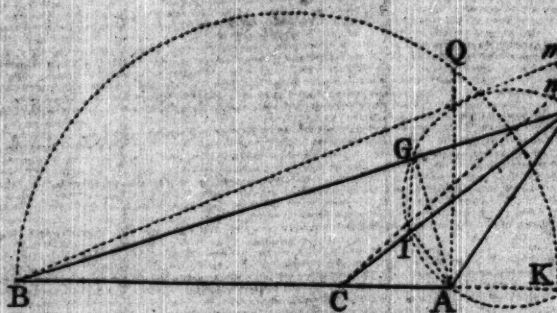


Fig. 15.

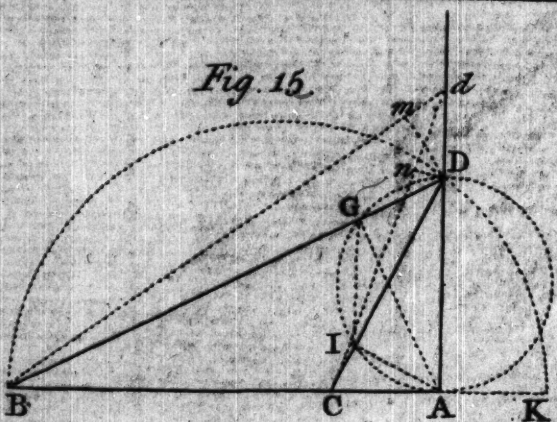


Fig. 16.

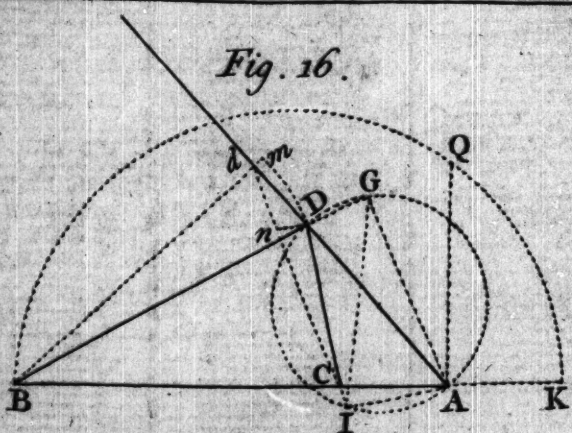
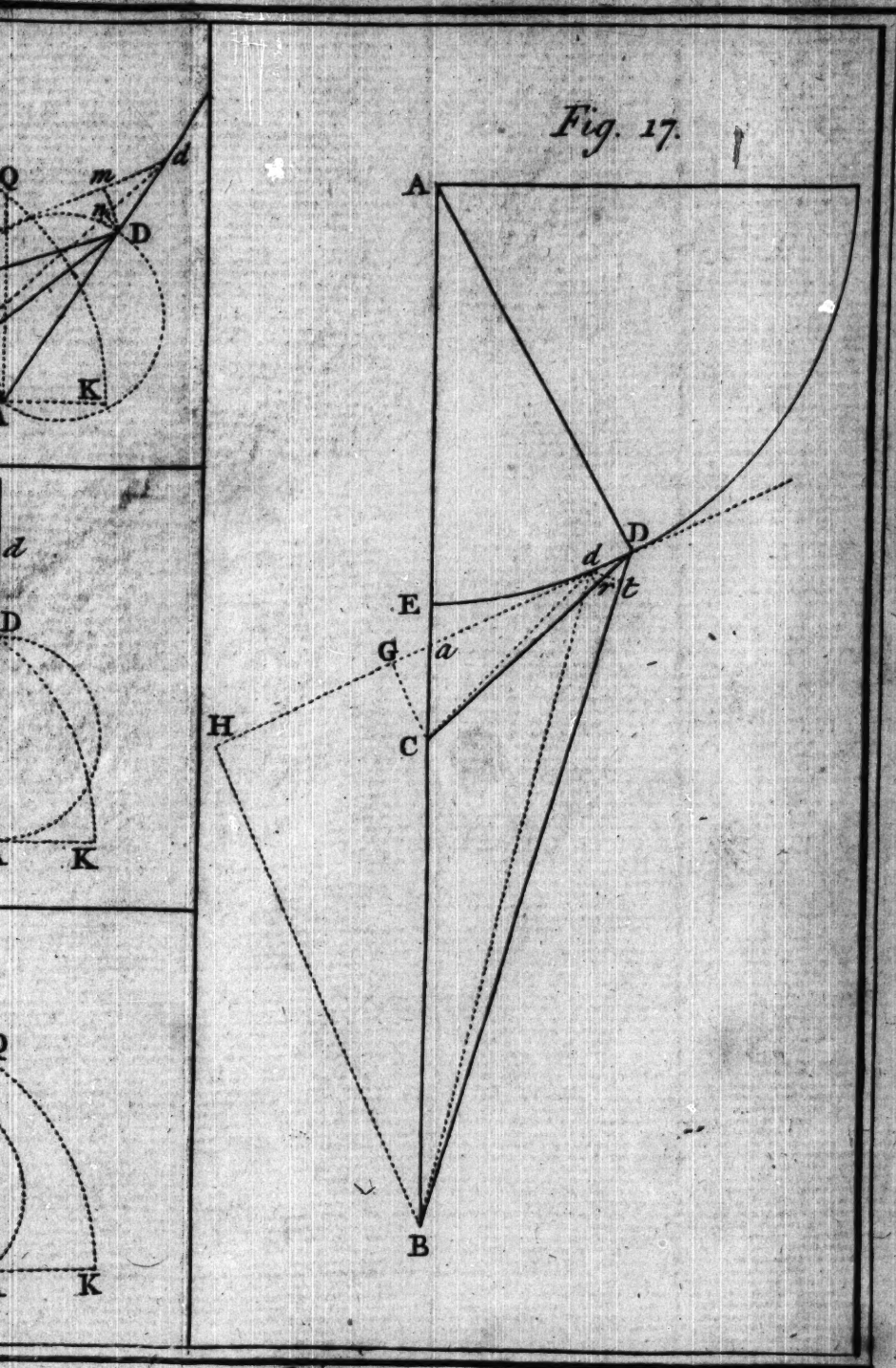


Fig. 14 to 17.

Fig. 17.



J. Mynde Sc.

Plate IV.

Fig. 18.

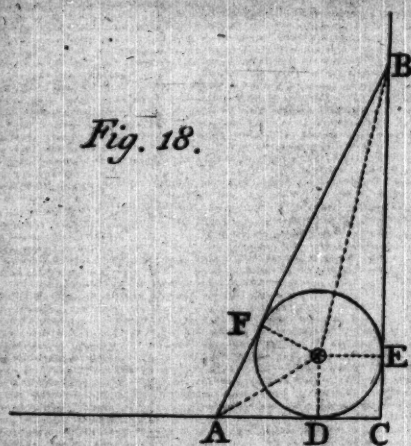


Fig. 21.

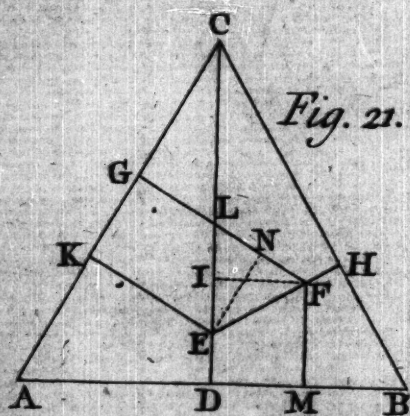


Fig. 22.

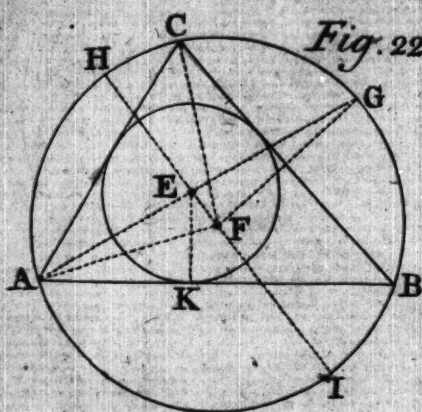


Fig. 18. to 24.

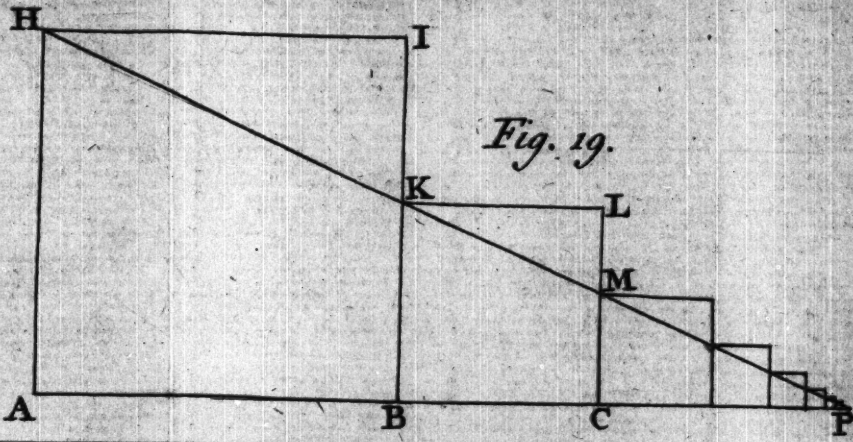


Fig. 19.

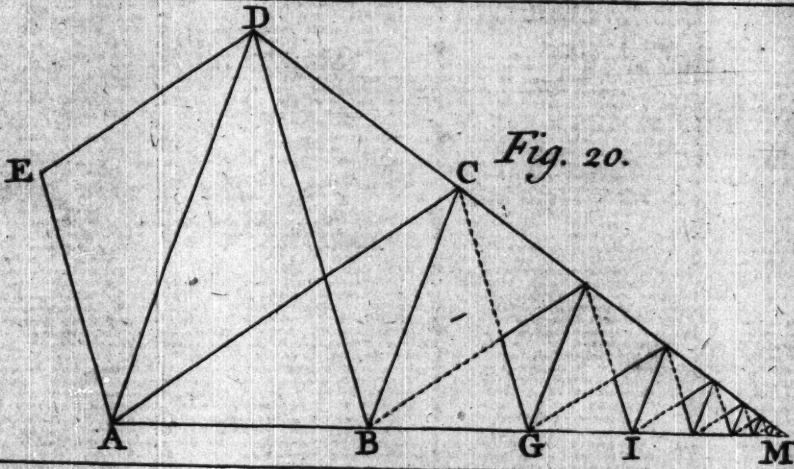


Fig. 20.

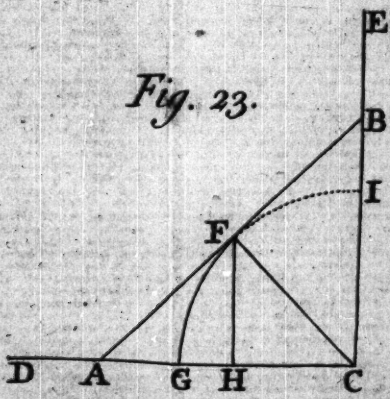


Fig. 23.

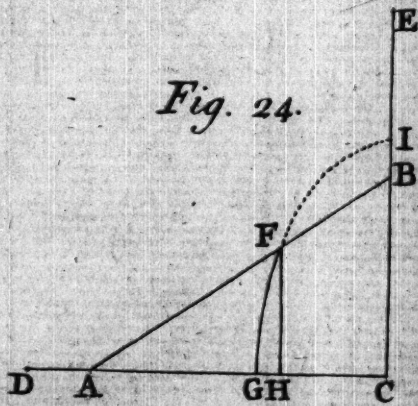


Fig. 24.

Plate V.

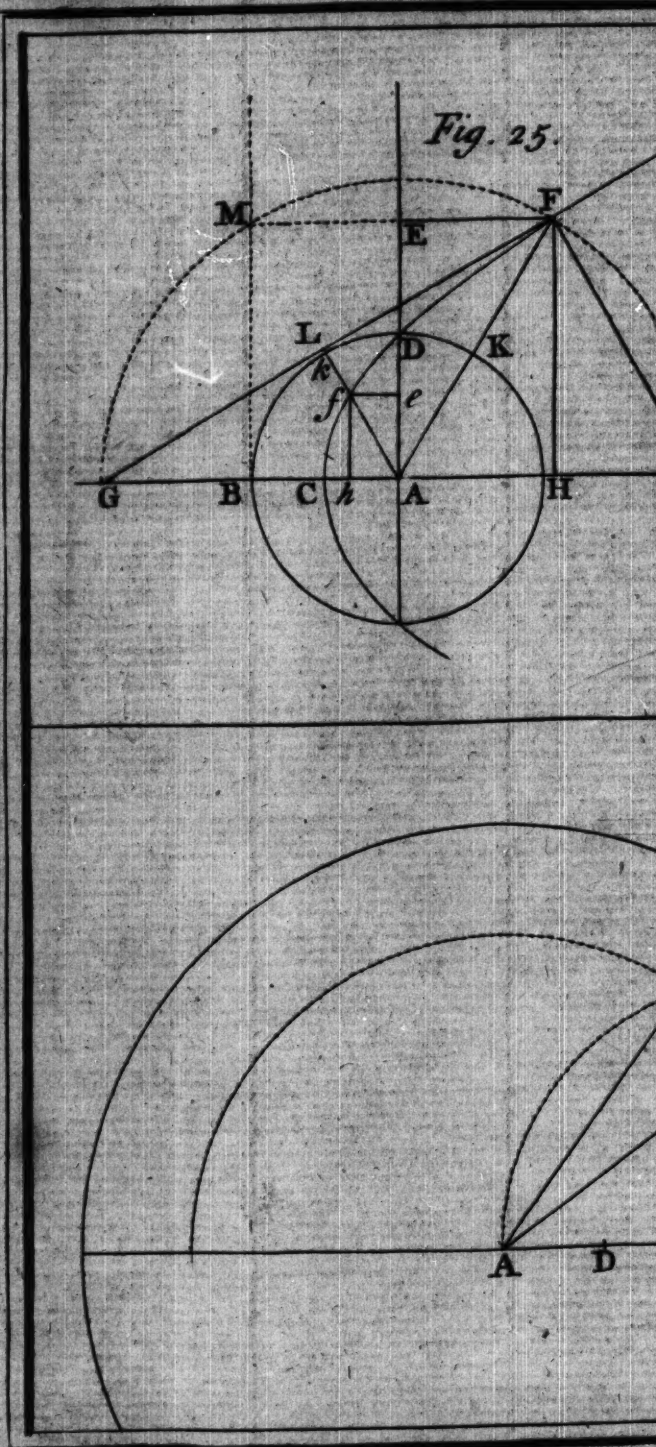


Fig. 25 to 27.

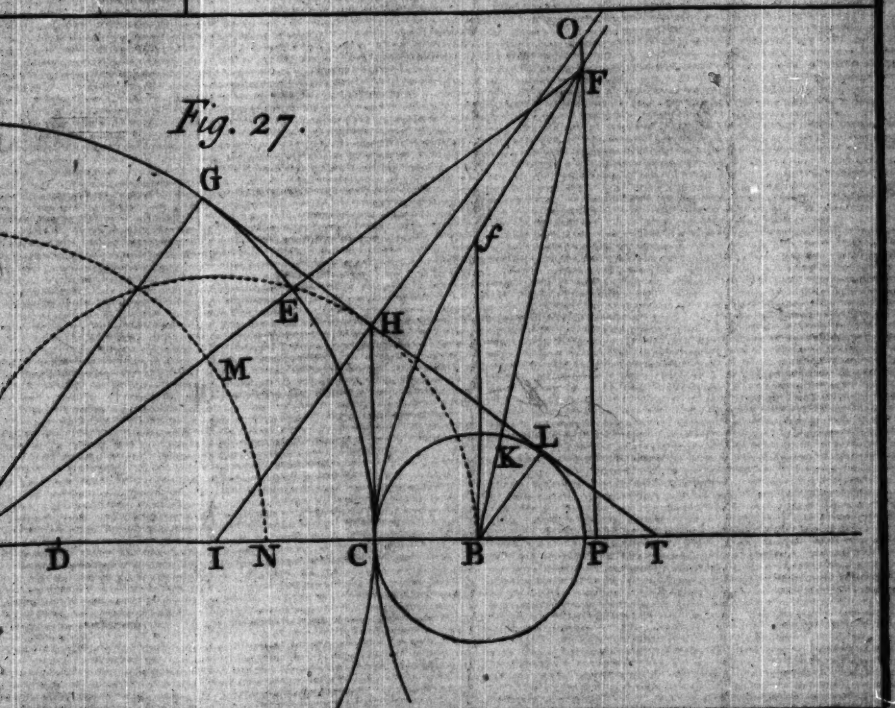
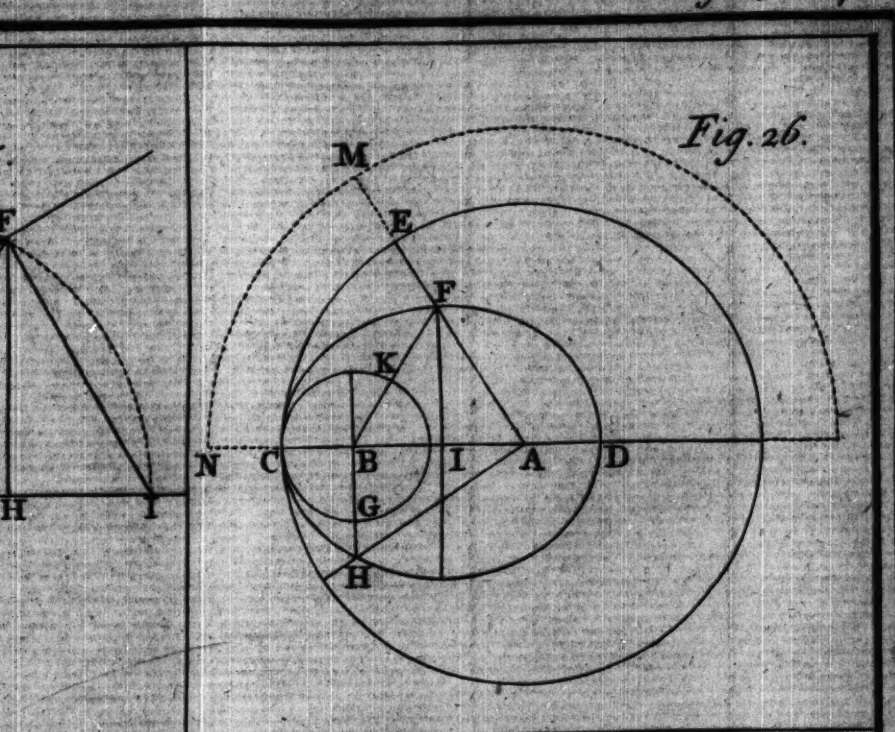


Plate VI.

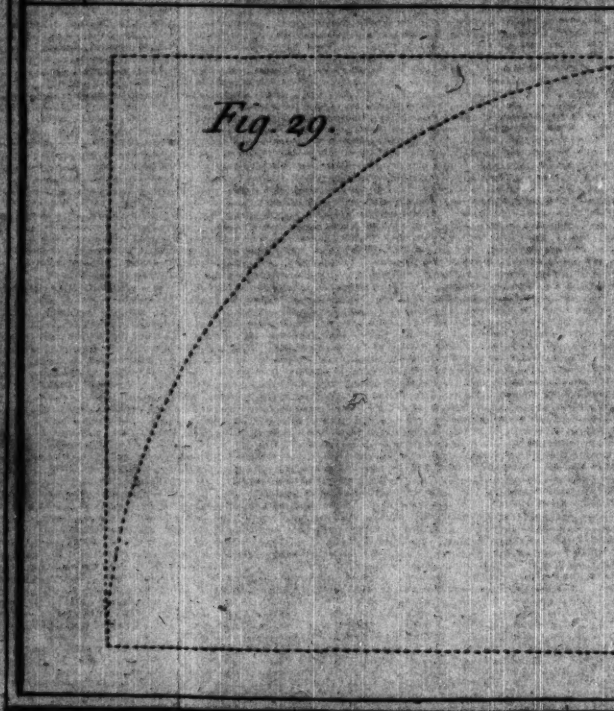
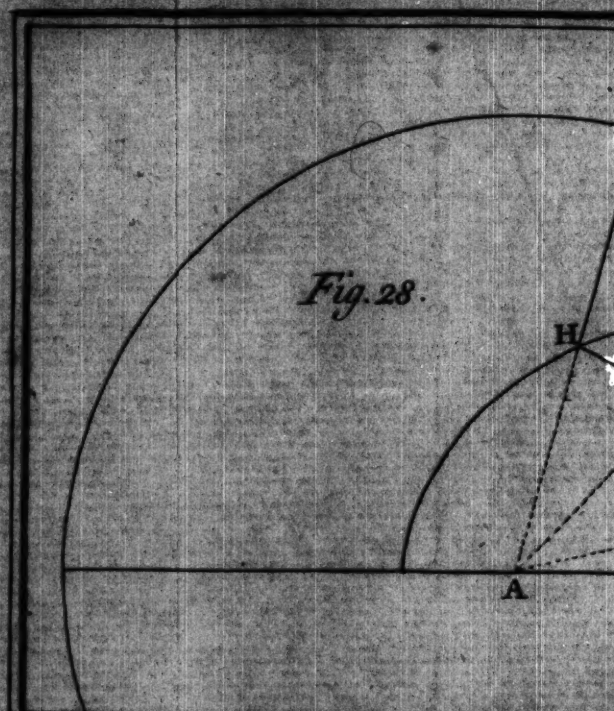


Plate VII

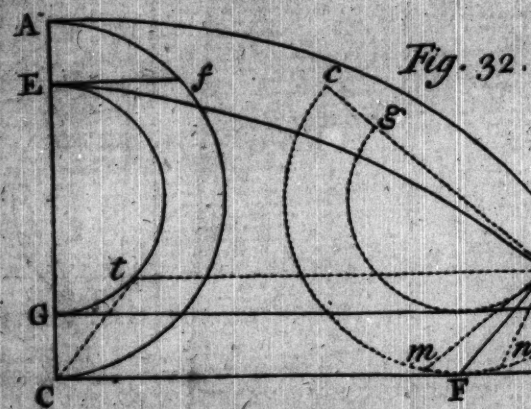
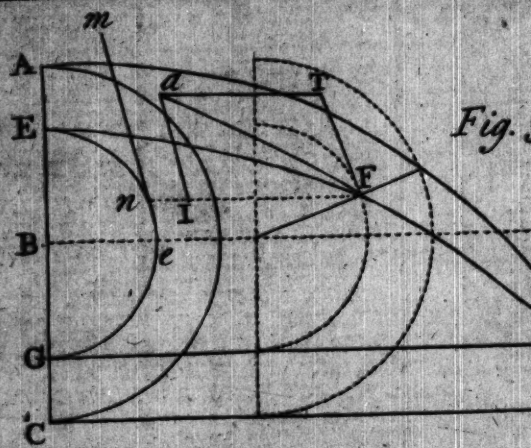
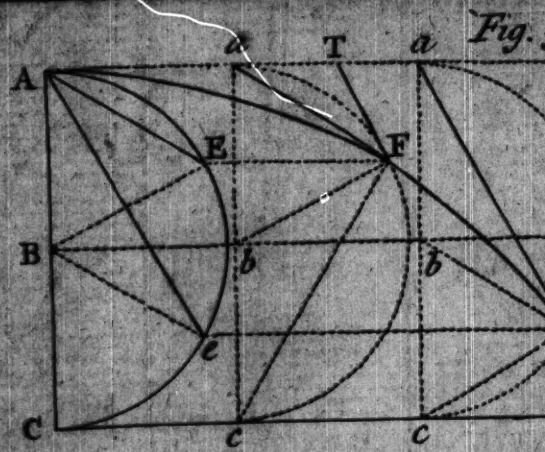


Fig. 30.



Fig. 33.

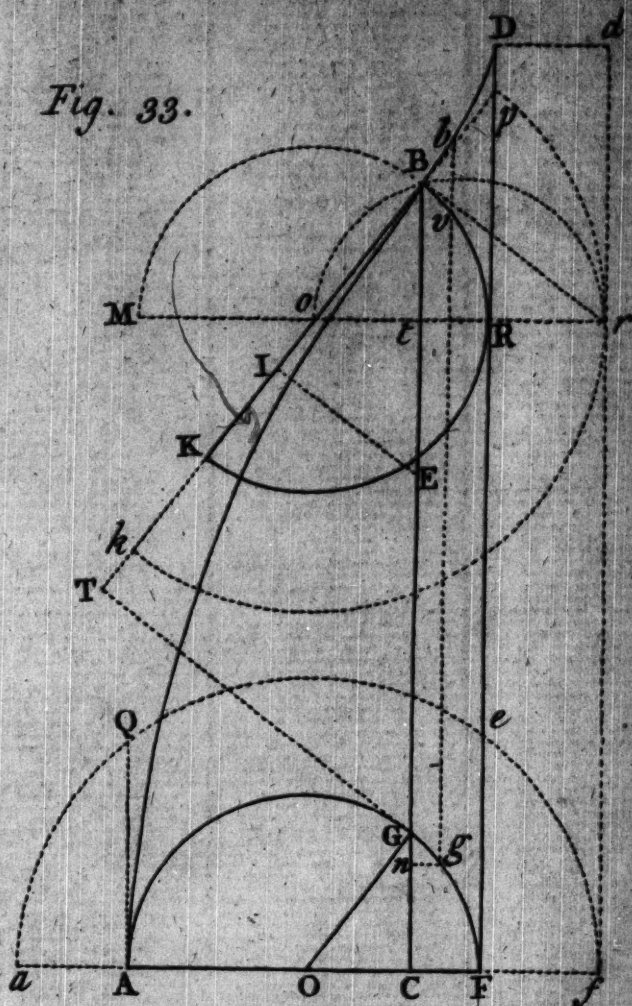


Fig. 31.

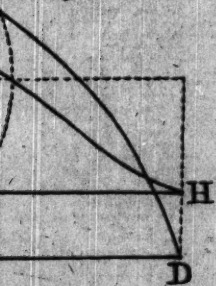


Fig. 32.

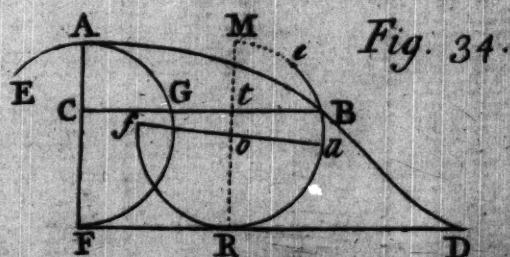
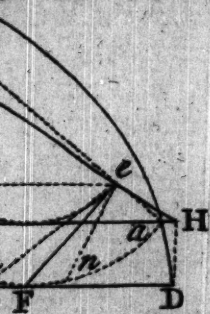
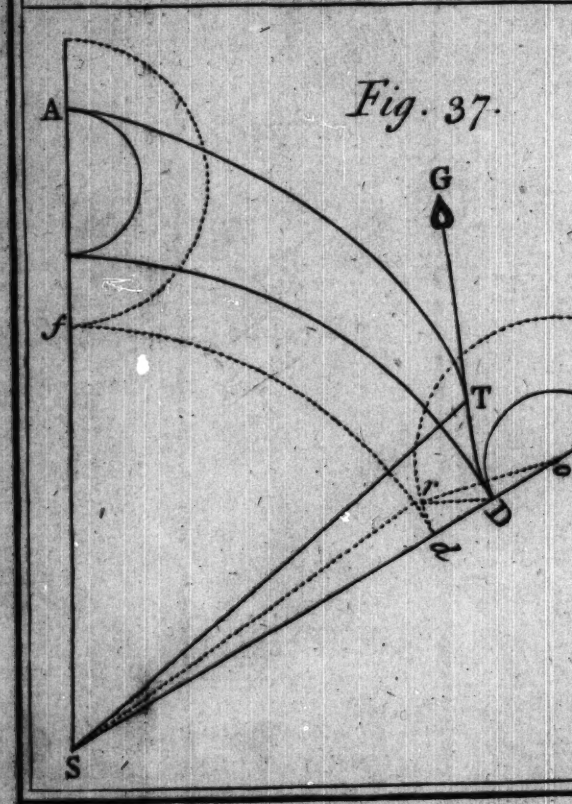
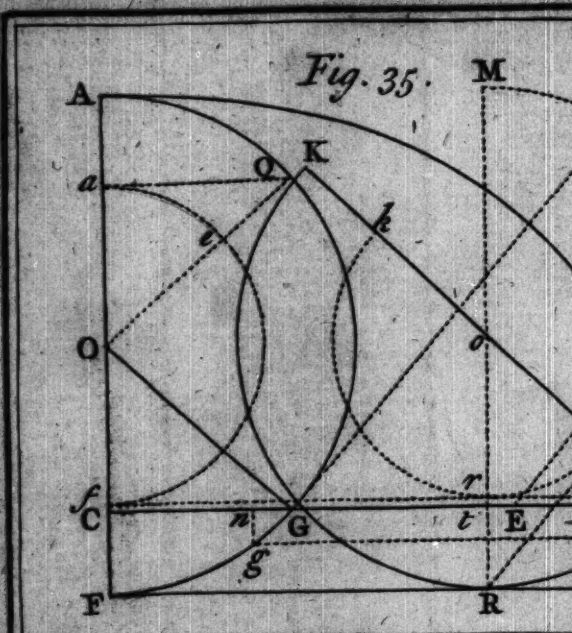


Plate VIII.



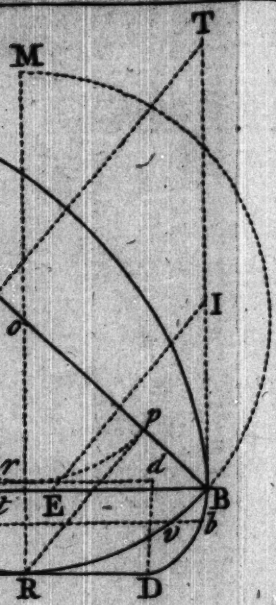
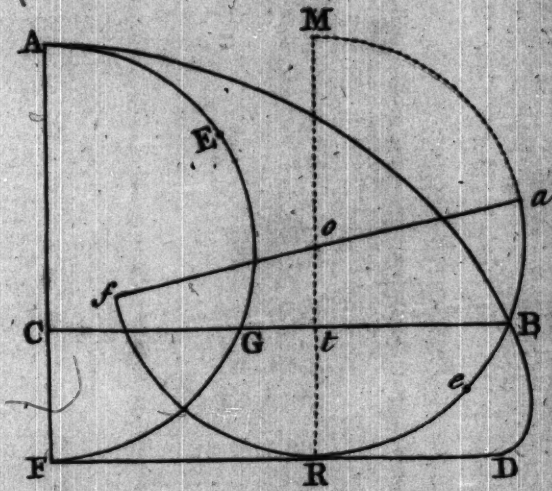


Fig. 36.



7.

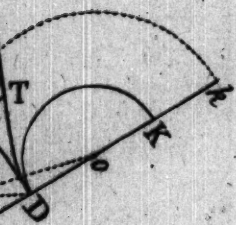


Fig. 38.

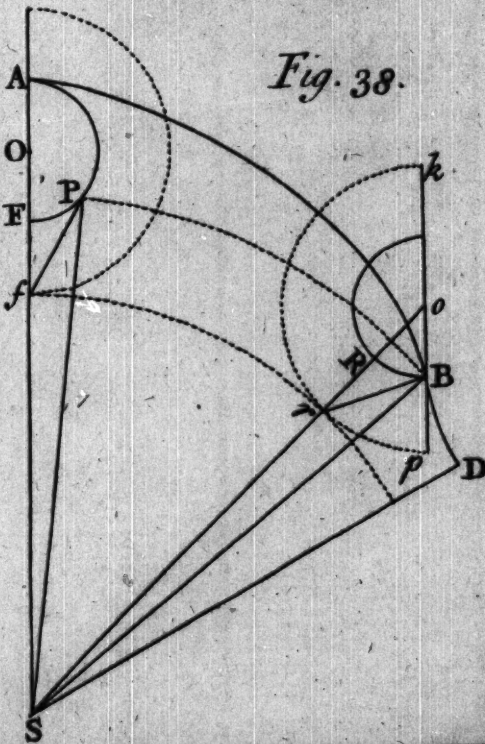


Plate IX.

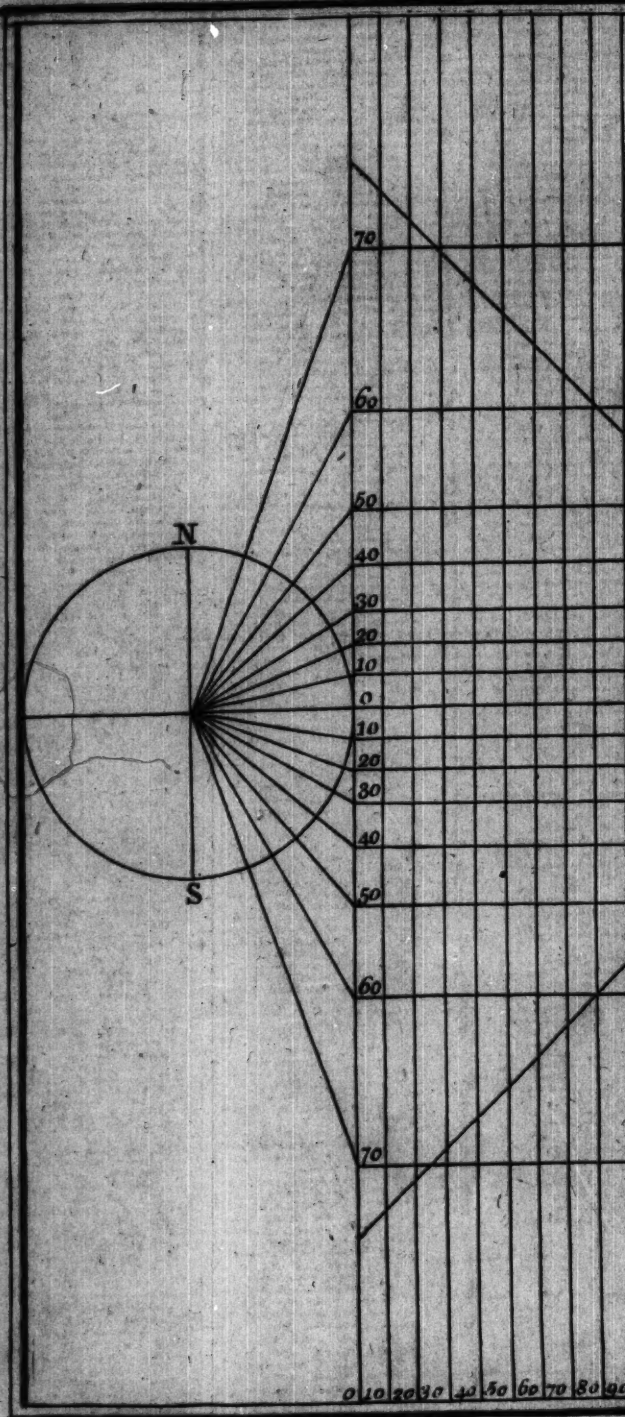
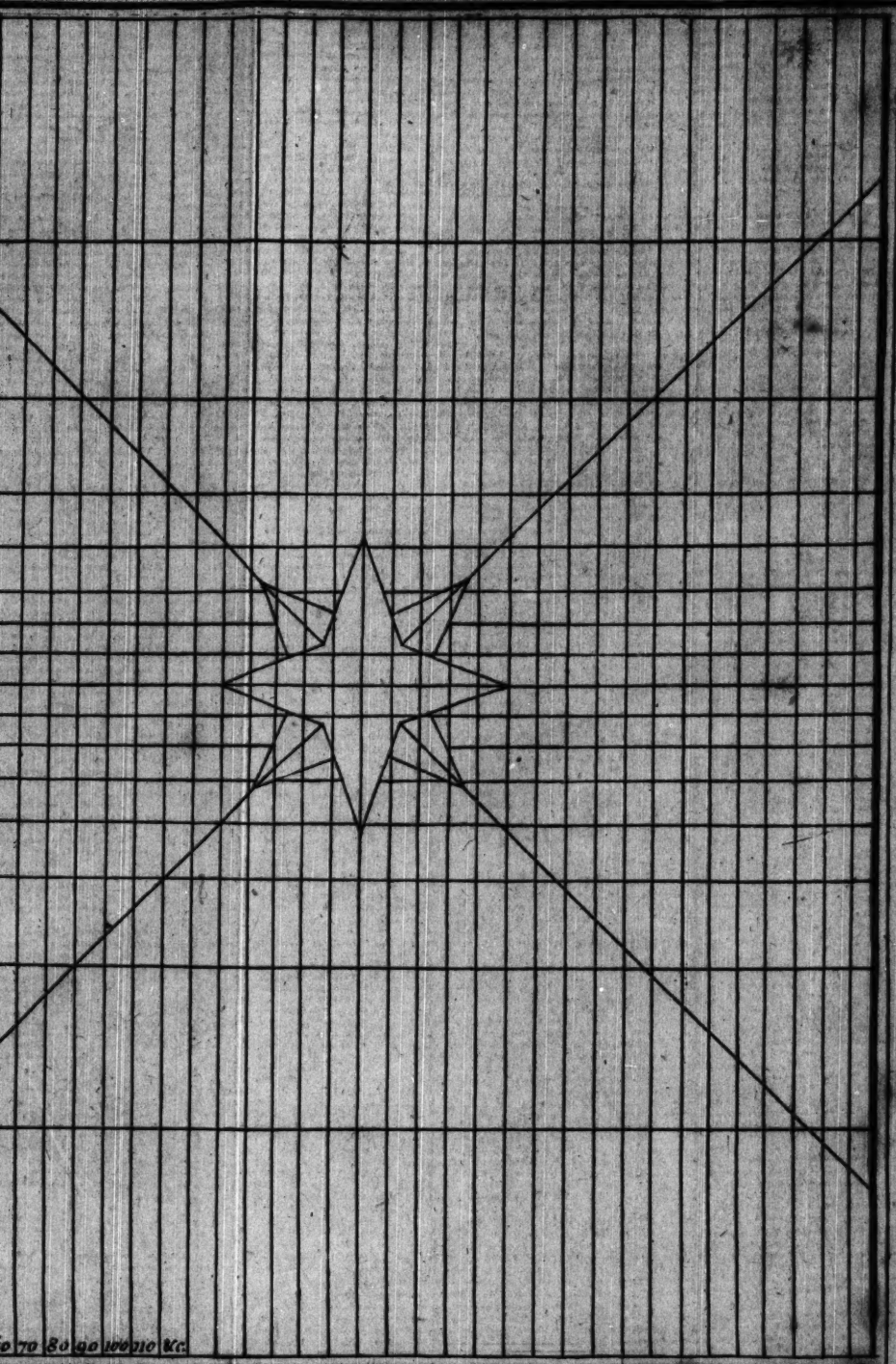


Fig. 39.



0 70 80 90 100 110 Kc.

L. Mynde Jr.

Plate X.

Fig. 40.

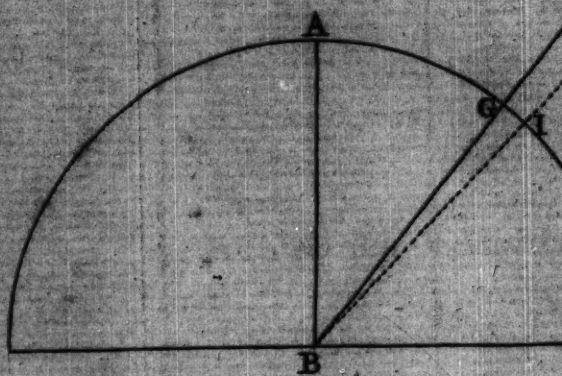


Fig. 41.

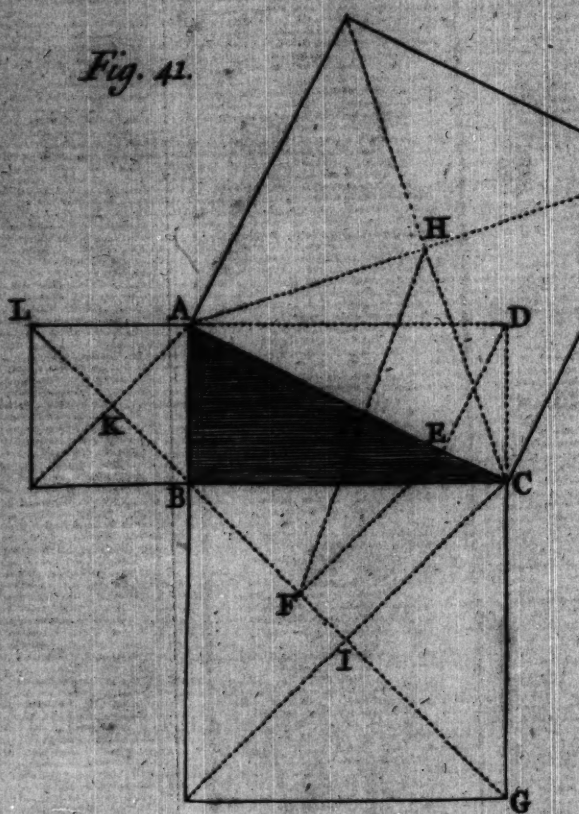


Fig. 40 to 43.

Fig. 42.

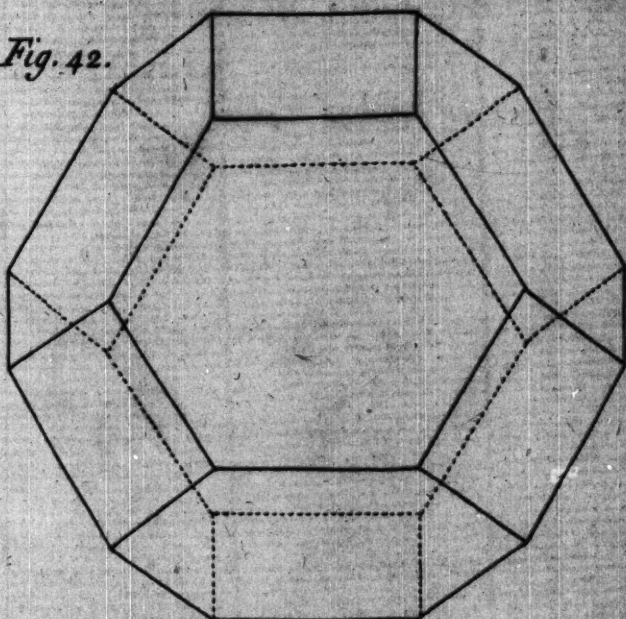
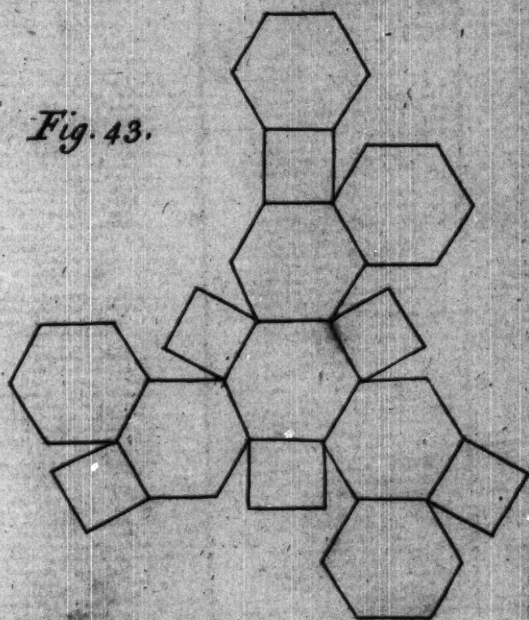


Fig. 43.



J. Mynde Jr.

Fig. 44.

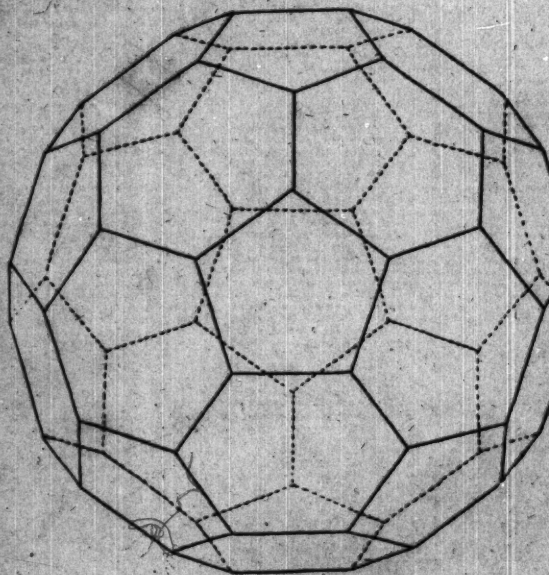


Fig. 46.

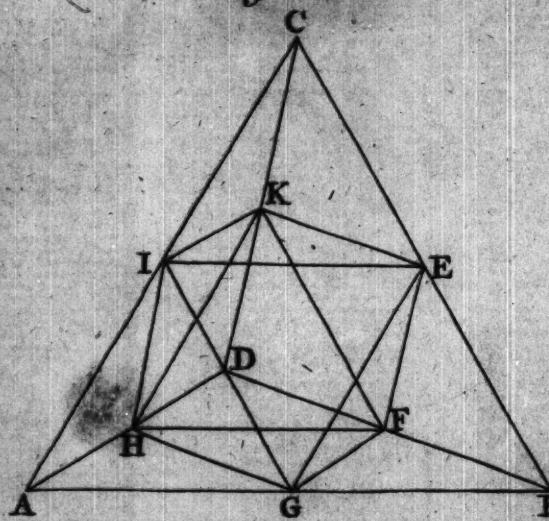
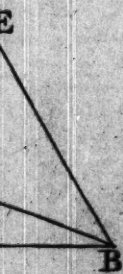
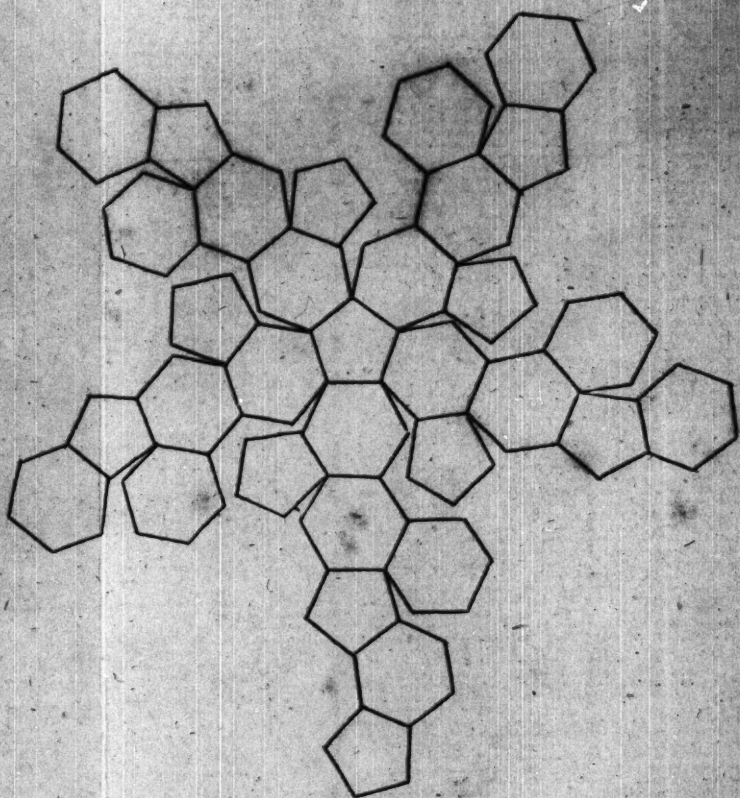


Fig. 44 to 46.

Fig. 45.



J. Mynd Sculpt